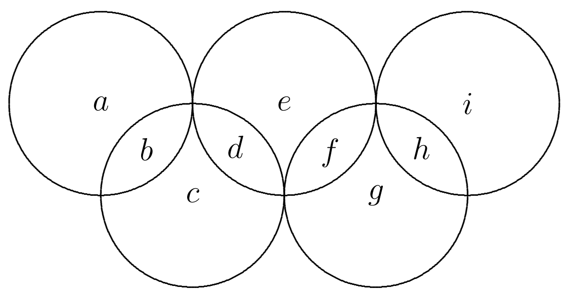
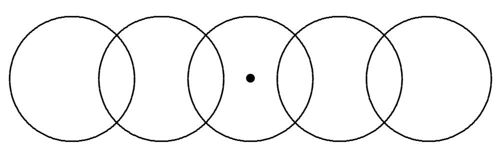
This essay is about magical figures. We're trying to distribute some positive integers into a geometric figure subject to certain conditions. Magic squares are the most common. Here you'll see several other designs. [1]

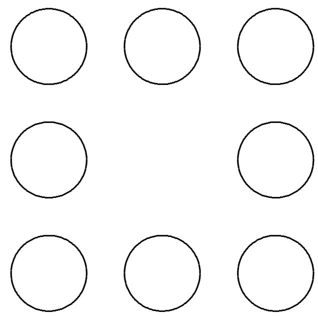
1. Place the numbers $1, 2, \ldots, 9$ in each of the nine regions (one number in each region) formed by the five circles as shown in the following figure so that the sum of the numbers inside each circle is 14.



2. Place the numbers $1, 2, \ldots, 9$ in each of the nine regions (one number in each region) formed by the five circles as shown in the following figure so that the sum of the numbers inside each circle is the same. What numbers goes in the region marked by the dot?



3. Consider the array of circles shown below.



(a) Distribute the members of $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ in the circles so that the four sums along horizontal and vertical lines are the same. Let K denote that constant line sum. Call K the *target value*.

(b) Can K be odd?

(c) For what values of K is there a solution?

(d) Among all solutions, what is the largest possible sum of the four corners?

(e) Now let $T = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Which elements of T can be removed so that the remaining eight element set of digits can be arranged to produce a magic square.

(f) Suppose $R = \{12, 18, 24, 31, 37, 43, 55, 61\}$. Can the members of R be magically distributed? Are there multiple solutions?

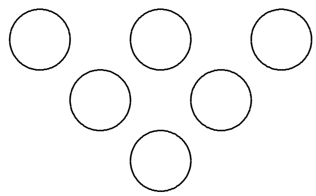
(g) Next suppose you have to make all the sums even or all the sums odd. Using 0 for even and 1 for odd, find all solutions. A good name for an odd-even solution is *pattern*. Find all patterns with an unequal number of 0's and 1's.

(h) Suppose that using all different numbers is not required. Then putting the same number in each position is a solution. Find a way of producing new solutions from old solutions using arithmetic operations. Note that we also need not require that the entries be whole numbers. If the entries can be any real numbers, the solutions have a very nice structure, something you have studied before. Find this structure.

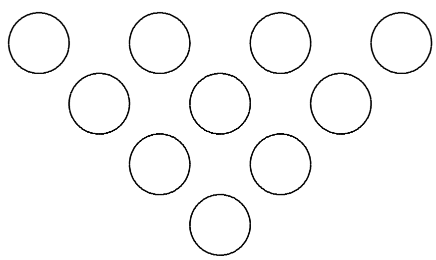
(i) Next we'll consider multiplication instead of addition. Can you distribute the eight members of S so that the product of the numbers along each line is the same. Find a set of eight different positive integers such that the products are the same. What is the smallest positive integer product?

4. The next problem is about magic subtraction triangles.

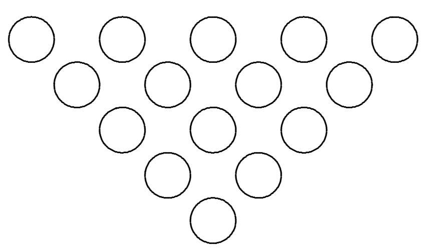
   (a) Put the numbers 1 through 6 in the circles so that each number in the bottom two rows is the positive difference between its two nearest neighbors above it.



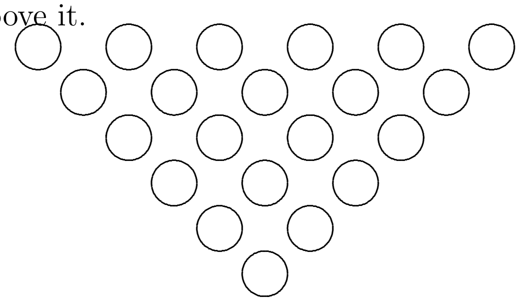
   (b) Put the numbers 1 through 10 in the circles so that each number in the bottom three rows is the positive difference between its two nearest neighbors above it.



   (c) Put the numbers 1 through 15 in the circles so that each number in the bottom four rows is the positive difference between its two nearest neighbors above it.

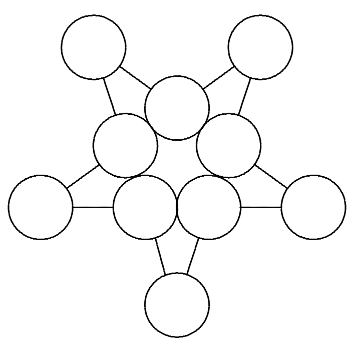


(d) Now try arranging the numbers 1 through 21 in the circles so that each number in the bottom five rows is the positive difference between its two nearest neighbors above it.



5. Look at the pentagram below. Notice that there are five 'lines' each pair of which intersect in a 'point'. Thus we have a finite *geometry*.

(a) Assign the numbers in the set $S = \{1, 2, 3, 4, 5, 6, 8, 9, 10, 12\}$ to the positions in the figure to make all five line sums the same.

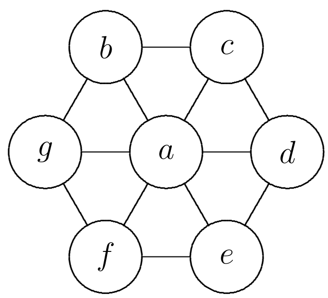


(b) Prove that it is impossible to distribute the integers $1, 2, 3, \ldots, 10$ in the ten circles so that the sums along the five lines of the pentagram are the same.

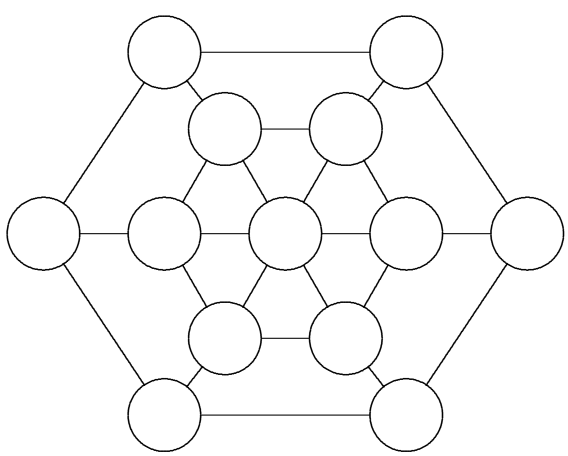
(c) Is there a ten-element subset of $\{1, 2, 3, \ldots, 11\}$ which admits such a distribution?

(d) Is there a ten-element subset of $\{1, 2, 3, \ldots, 12\}$ other than the one in problem 11 which admits such a distribution?

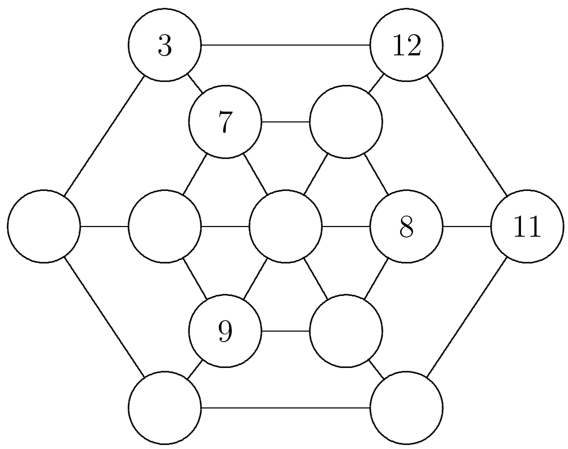
6. **Highly Unmagic Hexagram.** I'm grateful to Sam Vandervelde for this problem. Distribute the numbers 1 through 7 in the seven circles to that the nine lines $bc, cd, de, ef, fg, gb, abe, acf$ and adg all have **different** line sums.



7. Can you distribute the numbers 1 through 13 in the 13 circles so that the sums of the entries in the three lines though the center and the sums of the six entries in the two hexagons are identical? This problem, called the Magic Double Hexagon, was posed by Nadejda Dykevich in the New York Times Numberplay Blog by Gary Antonick during the week of July 17, 2013.



8. The original problem posed was a special case of problem 7 considered here. For completeness, the original problem is given below.



9. Can the numbers from 1 to 19 be placed one number in each position, in such a way that the sum of the entries on each of the hexagram's nine lines is the same?

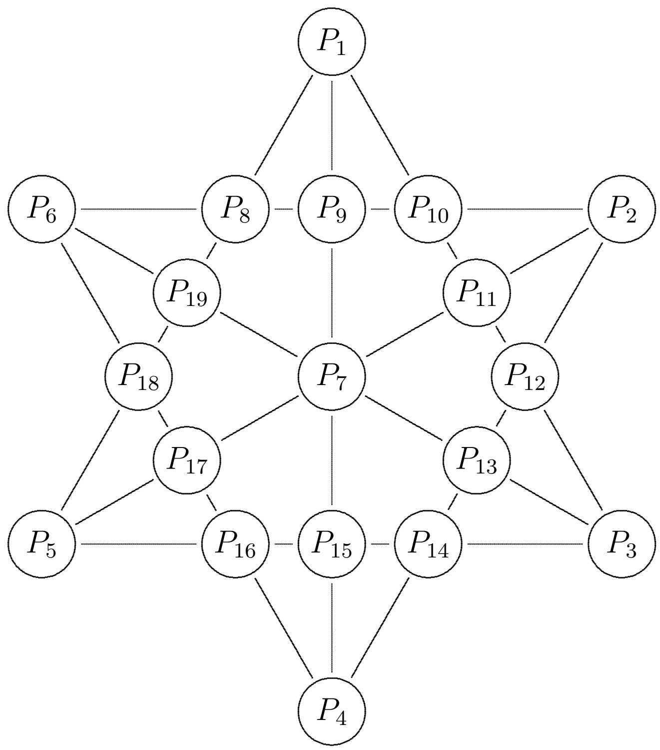
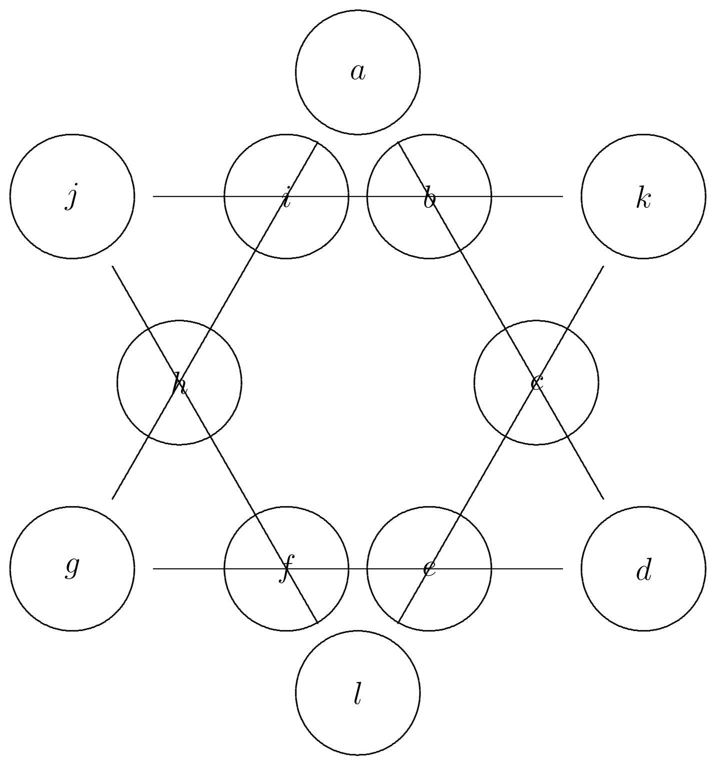


Figure 1.

10. Insert the numbers 1 though 12 in the circles so that the sums along the six lines are the same.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