1 Introduction

This essay introduces some new sets of numbers. Up to now, the only sets of numbers (algebraic systems) we know is the set $\mathbb{Z}$ of integers with the two operations $+$ and $\times$ and the system $\mathbb{R}$ of real numbers with the same two operations. Of course the operations satisfy lots of properties like commutativity, associativity, and distribution of $\times$ over $+$ (ie. $a(b+c) = ab + ac$. Both the systems $\mathbb{Z}$ and $\mathbb{R}$ are infinite. Our new systems are **finite**! There is one for each positive integer greater than 1. If n is such a positive integer, the notation for the new system is $\mathbb{Z}_n$. For convenience and simplicity, we are going to select just two values $n = 6$ and $n = 7$ to study in depth. [1]

2 Integer Division

We all know very well how to divide one positive integer by another. For example, if you were asked to compute 41/9, you might write $9\overline{)41}$ and proceed to carry out the division algorithm. Of course in doing this you would rely heavily on your knowledge of multiplication. It comes as no surprise to us that division requires some guesswork followed by multiplication to check the guesswork. Thus, you would say 9 goes into 41 4 times with 5 left over. We can express this as

$$41 = 9 \cdot 4 + 5.$$

In this equation, the number 41 is called the *dividend*, 9 is the *divisor*, 4 is the *quotient*, and 5 is the *remainder*. So, in general, we have

$$\textit{dividend} = \textit{divisor} \cdot \textit{quotient} + \textit{remainder}.$$

Using the letters D for dividend, d for divisor, q for quotient, and r for remainder, we write $D = d \cdot q + r$. Notice that our remainder 5 is in the range $0, 1, 2, 3, 4 \ldots, 8$. Why can we insist that this always happens when we are dividing by 9? What would a remainder larger than 9 mean? What would a negative remainder mean? If we insist, as we shall from here on that the remainder r be in the range $0, 1, 2, \ldots, d-1$, we may then assert the division algorithm:

Division Algorithm Given any integer D and any nonzero integer d, there exist unique integers q and r satisfying both

a. $D = d \cdot q + r$ and

b. $0 \leq r \leq |d| - 1$.

Note that in case $r = 0$, we call d a *divisor* of D. Notice also that neither D nor d is required to be positive. When either or both is negative, the equations and uniqueness still hold. For example, when $D = -31$ and $d = 9$, we get

$$-31 = 9(-4) + 5.$$

You'll see that this is very important when we use the division algorithm to find the representation of a positive or negative integer when the base number is negative.

Look at the two examples, $41 = 9 \cdot 4 + 5$, and $-31 = 9(-4) + 5$. In both cases we can say that q is the largest number such that $9q$ is less than D. Geometrically, think of marking of segments of length 9 units to the right if $D > 0$, and to the left if $D < 0$ until we get to the multiple of 9 that is 8 or fewer units to the left of D.

There is a nice way to write q and r in terms of D and d. To explore this, define the *floor* function $\lfloor x \rfloor$ of a number x as the largest integer that is not bigger than x. This useful function has a companion function, the *ceiling* function, $\lceil x \rceil$ which is the smallest integer that is not less than x. Another function we will need to know about is the fractional part function $\langle x \rangle$ which is defined as follows: $\langle x \rangle = x - \lfloor x \rfloor$. See the exercises and those in the module that covers place value.

a For each of the real numbers x listed, find $\langle x \rangle$, $\lfloor x \rfloor$ and $\lceil x \rceil$.

(a) $x = 3.14$

(b) $x = -3.14$

(c) $x = 4.216$

(d) $x = -4.216$

b Sketch the graphs of the three functions on the cartesian coordinate system.

c For each of the pairs D and d listed, find q and r using the formulas $q = \lfloor D/d \rfloor$ and $r = D - \lfloor D/d \rfloor \cdot d$. Notice that your graphing calculator has the functions $\langle x \rangle$, $\lfloor x \rfloor$ and $\lceil x \rceil$ built in.

(a) $D = 77, d = 5$

(b) $D = -77, d = 5$

(c) $D = 77, d = -5$

(d) $D = -77, d = -5$

3 Modulo 9 congruence

Note that both $41 = 36 + 5$ and $-31 = -36 + 5$ are 5 units larger than a multiple of 9. Note that their difference $41 - (-31) = 72$ is a multiple of 9. Of course this happens for any two numbers D_1 and D_2 whenever the remainders, upon division by 9, are the same. To prove this, suppose r is that remainder. Then

$$D_1 = 9 \cdot q_1 + r \quad \text{and} \quad D_2 = 9 \cdot q_2 + r.$$

Then

$$\begin{aligned} D_1 - D_2 &= 9 \cdot q_1 + r - (9 \cdot q_2 + r) \\ &= 9 \cdot q_1 + r - (9 \cdot q_2 + r) \\ &= 9 \cdot q_1 - 9 \cdot q_2 + r - r \\ &= 9 \cdot (q_1 - q_2). \end{aligned}$$

On the other hand, the converse is also true: If two numbers D_1 and D_2 differ by a multiple of 9, upon division by 9, the remainders are the same. To see this, suppose $D_1 = 9 \cdot q_1 + r_1 \quad \text{and} \quad D_2 = 9 \cdot q_2 + r_2$. Since $D_1 - D_2$ is a multiple of 9, we can write $D_1 - D_2 = 9 \cdot q_1 + r_1 - (9 \cdot q_2 + r_2) = 9k$, for some integer k. We can solve this for $r_1 - r_2$ to get $r_1 - r_2) = 9k - (9 \cdot q_1 - 9 \cdot q_2) = 9(k - q_1 + q_2)$, so $r_1 - r_2$ is a multiple of 9. Since both r_1 and r_2 are in the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8\}$, the largest they could differ by is 8 and the least they could differ by is -8. The only multiple of 9 in that range is 0, so $r_1 - r_2 = 0$, which is what we wanted to prove.

We encounter two integers D_1 and D_2 that have this relationship often enough to give the relationship a special name: *modular congruence*. Two numbers that differ by a multiple of 9 are said to be congruent modulo 9, and we write $D_1 \equiv D_2 \pmod 9$.

Of course there is nothing special about the number 9 except that it is positive. Hence we have the following definition.

Congruence Modulo m Let m be a positive integer. Two numbers D_1 and D_2 are congruent modulo m, and we write $D_1 \equiv D_2 \pmod m$ if $D_1 - D_2$ is a multiple of m.

The definition of congruence, which has nothing to do with the same word used in geometry, is due to Carl Gauss. Gauss noticed that the relationship that two numbers D_1 and D_2 have the same remainder upon division by another number m

kept turning up in his examples and proofs. For example, the set of even numbers are those which yield a zero remainder when divided by 2. Thus congruence modulo 2 means that the two integers are either both even or both odd. This is a good time to practice with congruences.

4 Properties of Modular Congruence

In this section, we discuss three important properties of modular congruence and in the next section, two more. The first three are the properties required for an equivalence relation. An *equivalence* relation is a binary relation that satisfies the three properties of *reflexivity*, *symmetry*, and *transitivity*, defined below. In the Discrete Math course, you will learn all about binary relations. However, we do not assume here that you are familiar with the properties of binary relations.

1. Congruence modulo m is a *reflexive* relation. That is, for every integer u, $u \equiv u \pmod{m}$. This is clearly true because $u - u = 0$ is a multiple on m.

2. Congruence modulo m is a *symmetric* relation. This means that if u and v are integers satisfying $u \equiv v \pmod{m}$, then $v \equiv u \pmod{m}$. This property follows from the fact that the negative of a multiple of m is also a multiple of m.

3. Congruence modulo m is a *transitive* relation. This means that for any three integers u, v and w satisfying $u \equiv v \pmod{m}$ and $v \equiv w \pmod{m}$, it is also true that $u \equiv w \pmod{m}$. To prove this, let u, v and w satisfy the hypothesis. Then $u - v = km$ and $v - w = lm$, where k and l are integers. Now $u - w = u - v + v - w = km + lm = (k + l)m$. In other words, $u - w$ is a multiple of m.

Of course you recognize that equality (=) satisfies all three of these, and indeed, modular congruence **is** a relation very much like equality. Besides discrete math, you might see *equivalence relations* in the course that develops the number system and in the abstract algebra course. Relations satisfying properties 1, 2, and 3, that is, equivalence relations, above are common in mathematics. The most important property of equivalence relations is that each one gives rise to a partition of the set on which it is defined. The sets in the partition are called the *cells* of the relation. See theorem 1 below.

In discussing the next two properties, we'll consider again the special case $m = 9$. Although the properties are true for any positive integer m, we are going use the case $m = 9$ repeatedly in the problems, and the proofs are simplified a tiny bit using the 9 where we otherwise use m.

5 Definition of Cells

Recall that we are discussing the relation that has a pair (u, v) (or u is related to v) provided $u \equiv v \pmod 9$, or equivalently, $u - v$ is a multiple of 9. For each integer k, the sell of $\overline{k}$, denoted k is the set of integers n to which k is related. Symbolically, $\overline{k} = \{n \mid k \equiv n \pmod 9\}$. Thus, for example,

$$\overline{0} = \{n \mid 0 \equiv n \pmod 9\} = \{0, \pm 9, \pm 18, \ldots\}$$

and

$$\overline{1} = \{n \mid 1 \equiv n \pmod 9\} = \{1, -8, 10, -17, 19, \ldots\}.$$

Notice that $\overline{0} = \overline{9}$ because any number that 0 is congruent to, 9 is also congruent to. To be a little more formal about about it, take an arbitrary member n of $\overline{0}$. Then $0 \equiv n \pmod 9$. Since $9 \equiv 0 \pmod 9$ and congruence modulo 9 is known to be transitive (see property 3 above), we see that the pair $9 \equiv 0 \pmod 9$ and $0 \equiv n \pmod 9$ implies $9 \equiv n \pmod 9$. So we have proved that if $n \in \overline{0}$ then $n \in \overline{9}$, which in set terms means $\overline{0} \subseteq \overline{9}$. Similarly, we can prove that $\overline{9} \subseteq \overline{0}$. But these two set inequalities are together equivalent to saying $\overline{0} = \overline{9}$.

6 The Partition of an Equivalence Relation

In this section we state and prove the theorem mentioned above on partitions induced by equivalence relations. Although a much more general theorem on equivalence relations is provable here, we will simply use the congruence modulo m on the set Z of integers. We proved Theorem 1 in section 3.

Theorem 1. Let m be a positive integer. Congruence mod m is an equivalence relation.

Theorem 2. Let m be a positive integer. For each integer u let $[u]$ denote the cell of u. If $[u]$ and $[v]$ are two cells, modulo m, then either $[u]$ and $[v]$ are disjoint or identical.

Proof. These cells are also called equivalence classes. To see that two cells $[u]$ and $[v]$ are identical if have any integers in common, note that if w belongs to both $[u]$ and $[v]$, then $u \equiv w \pmod{m}$ and $v \equiv w \pmod{m}$, by the definition of cell. By symmetry $w \equiv v \pmod{m}$ and by transitivity $u \equiv v \pmod{m}$. Then again symmetry by transitivity, any integer z in $[u]$ satisfies $z \equiv u \pmod{m}$ and hence $z \equiv v \pmod{m}$, so z belongs to $[v]$.

The Arithmetic of Congruences In order to build a mathematical structure using cells as "numbers", we have to be sure that addition and multiplication are well-defined operations. The method for defining the sum $\oplus$ and the product $\odot$ of two cells is pretty clear. For example, if $[x]$ and $[y]$ are cells modulo 6 we want $[x] \oplus [y]$ to be $[x + y]$ and $[x] \odot [y]$ to be $[x \cdot y]$. But we must be sure that these operations are well-defined. That is, $\oplus$ and $\odot$ must deliver unique values.

Theorem 3. Let x and y be integers and $[x]$ and $[y]$ their cells modulo m. Suppose $a \in [x]$ and $b \in [y]$. Then $a + b \in [x + y]$. In other words, no matter which representatives of $[x]$ and $[y]$ we choose, the sum is always the cell $[x + y]$.

Proof. To say $a \in [x]$ means that $x \equiv a \pmod{m}$. Likewise, $y \equiv b \pmod{m}$.

Now $x + y - (a + b) = (x - a) + (y - b)$ is the sum of two multiples of m, and therefore is itself a multiple of m. In other words $a + b \in [x + y]$, as we set out to prove.

Here is a nice application of Theorem 3.

Problem Find the units digit of the Fibonacci number F_{2000}.

Theorem 3 makes this easy. It says that the units digit of the sum of two F's is the units digit of the sum of the units digits of those F's. The table below shows the units digits of the first 60 Fibonacci numbers in groups of 10. The first 30 entries are given by

1	1	2	3	5	8	3	1	4	5
9	4	3	7	0	7	7	4	1	5
6	1	7	8	5	3	8	1	9	0

Notice that the next 30 entries are

9	9	8	7	5	2	7	9	6	5
1	6	7	3	0	3	3	6	9	5
4	9	3	2	5	7	2	9	1	0

We are now in position to build the table for $\oplus$ modulo m. Let's choose $m = 6$ again. In the exercises you'll get to do this for $m = 7$. Since division by 6 produces the 6 remainders 0,1,2,3,4, and 5, we use these numbers as names of cells. What is $[3] \oplus [5]$? To see this, divide $3+5$ by 6, getting a remainder of 2, so $[3] \oplus [5] = [3+5] = [2]$.

$\oplus$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[1]	[2]	[3]	[4]	[5]
[1]	[1]	[2]	[3]	[4]	[5]	[0]
[2]	[2]	[3]	[4]	[5]	[0]	[1]
[3]	[3]	[4]	[5]	[0]	[1]	[2]
[4]	[4]	[5]	[0]	[1]	[2]	[3]
[5]	[5]	[0]	[1]	[2]	[3]	[4]

Theorem 4. Let x and y be integers and let $[x]$ and $[y]$ be their cells modulo m. Suppose $a \in [x]$ and $b \in [y]$. Then $a \cdot b \in [x \cdot y]$. In other words $[x] \odot [y] = [x \cdot y]$ is a well-defined binary operation.

Proof. As before, the hypothesis means that $x - a$ and $y - b$ are multiples of m. Let's say $x - a = km$ and $y - b = lm$. Then $xy - ab = xy - ay + ay - ab = (x-a)\,y + a\,(y-b) = kmy + alm = m\,(ky + al)$ which, of course, is a multiple of m since $ky + al$ is an integer.

We can now build the $\odot$ for any positive integer m. Again we use $m = 6$.

$\odot$	[0]	[1]	[2]	[3]	[4]	[5]
[0]	[0]	[0]	[0]	[0]	[0]	[0]
[1]	[0]	[1]	[2]	[3]	[4]	[5]
[2]	[0]	[2]	[4]	[0]	[2]	[4]
[3]	[0]	[3]	[0]	[3]	[0]	[3]
[4]	[0]	[4]	[2]	[0]	[4]	[2]
[5]	[0]	[5]	[4]	[3]	[2]	[1]

The next theorem follows easily from Theorem 4. We'll use it to find remainders when the n^{th} power of an integer k is divided by another integer m.

Theorem 5. If $a \equiv b \pmod{m}$ then, for every positive integer n, $a^n \equiv b^n \pmod{m}$.

The proof is by mathematical induction on n. Skip ahead if you need to, to see how proofs by induction work. First note that $a^1 \equiv b^1 \pmod{m}$, the base case. Next, suppose $a^{n-1} \equiv b^{n-1} \pmod{m}$. Now apply Theorem 2 with $x \equiv b$ and $y = b^{n-1}$. Then $a \equiv b \pmod{m}$ and $a^{n-1} \equiv b^{n-1} \pmod{m}$. Therefore $a \cdot a^{n-1} \equiv b \cdot b^{n-1} \pmod{m}$, and we are done.

The next example is of the type promised just before Theorem 5.

Problem Find the remainder when 7^{2009} is divided by 5.

7 Divisibility by 9, 3 and 11

The aim of this section is to demonstrate an easy method for finding the cell of a positive integer written in decimal notation modulo 3 or 9 or 11. by the way, the cell is often referred to as the residue class of the number. For example, we could say that the residue class of 101 modulo 9 is 2 because you get a remainder of 2 when you divide 101 by 9.

First we'll take an example. What is the residue class of 3742 modulo 9? Notice that $1000 \equiv 999 + 1 \equiv 0 + 1 \equiv 1 \pmod 9$. Likewise, $100 \equiv 99 + 1 \equiv 0 + 1 \equiv 1 \pmod 9$. Of course $10 \equiv 1 \pmod 9$ also. Using the language of modular arithmetic, we have $3742 \equiv 3000 + 700 + 40 + 2 \equiv 3 \cdot 1000 + 7 \cdot 100 + 4 \cdot 10 + 2 \equiv 3 \cdot 1 + 7 \cdot 1 + 4 \cdot 1 + 2 \equiv 3 + 7 + 4 + 2 \equiv 7 \pmod 9$. Thus 3742 is congruent modulo 9 to the sum of its digits.

Theorem 6. Every positive integer n is congruent modulo 9 to the sum of its decimal digits.

Proof. Recall that the decimal representation of a number is a sum of multiples of powers of 10. Thus

$$\begin{aligned} n &= (a_k \dots a_0)_{10} \\ &= a_k \left(\underbrace{99\dots9}_{\text{k 9's}} + 1 \right) + a_{k-1} \left(\underbrace{9\dots9}_{\text{(k-1) 9's}} + 1 \right) + \dots + a_1 (9+1) + a_0 \\ &= a_k \left(\underbrace{99\dots9}_{\text{k 9's}} \right) + a_{k-1} \left(\underbrace{9\dots9}_{\text{(k-1) 9's}} \right) + \dots + a_1 (9) + (a_k + a_{k-1} + \cdots a_0) \\ &= 9M + \left(\underbrace{a_k + a_{k-1} + \cdots + a_0}_{\text{sum of digits}} \right). \end{aligned}$$

Notice that the very same reasoning works for congruence modulo 3. Modulo 11 work is a bit more complicated. Note that $1 \equiv 1 \pmod{11}$, $10 \equiv -1 \pmod{11}$, $100 = 10^2 \equiv (-1)^2 \equiv 1 \pmod{11}$. In general $10^n \equiv (-1)^n \pmod{11}$. Thus $3742 \equiv 2 - 4 + 7 - 3 \equiv 2 \pmod{11}$. In other words, when we divide 3742 by 11, we get a remainder of 2. The sum $2 - 4 + 7 - 3$ is called the *alternating sum* of digits.

8 Problems and Exercises

8.1 SET 1

1. Find the congruence class modulo 9 of each of the given integers.
   (a) 12345
   (b) 637228195
   (c) 12345678910111213 . . . 99 obtained by writing down the positive integers from 1 to 99 next to one another.
2. Find the congruence class modulo 3 of each of the given integers.
   (a) 12345
   (b) 637228195
   (c) 12345678910111213 . . . 99 obtained by writing down the positive integers from 1 to 99 next to one another.
3. Find the congruence class modulo 11 of each of the given integers.
   (a) 12345
   (b) 637228195
   (c) 12345678910111213 . . . 99 obtained by writing down the positive integers from 1 to 99 next to one another.
4. Find the congruence class of each of the integers given modulo 99.
   (a) 12345
   (b) 637228195
   (c) 12345678910111213 . . . 99 obtained by writing down the positive integers from 1 to 99 next to one another.
5. Find the congruence class of each of the integers given modulo 66.
   (a) 12345
   (b) 637228195
   (c) 12345678910111213 . . . 99 obtained by writing down the positive integers from 1 to 99 next to one another.

8.2 SET 2

6. Build the $\oplus$ and $\odot$ tables for the congruence classes modulo 7. The mathematical system we get is denoted $(Z_7, \oplus, \odot)$.

   (a) show that $(Z_7, \oplus)$ is a group.

   (b) show that $(Z_7, \{[0]\}, \odot)$ is a group.

   (c) $(Z_7, \oplus, \odot)$ has the other properties (field axioms). This proves that $(Z_7, \oplus, \odot)$ is a field. You'll study structures like this in the abstract algebra course.

   (d) Solve the linear congruences.
   $[2] \odot [x] + [1] = [0]$
   (some texts would write this $2x + 1 = 0$.)

7. Among the integers $1, 2, \ldots, 2008$, what is the maximum number of integers that can be selected such that the sum of any two selected number is not a multiple of 7.

8. There are three prime numbers whose squares sum to 5070. What is the product of these three numbers?

9. Find the remainder when 333^{333} is divided by 33.

10. There is a pile of eggs. Joan counted the eggs, but her count was of by 1 in the 1's place. Tom counted the eggs, but his count was of by 1 in the 10's place. Raoul counted the eggs, but his count was of by 1 in the 100's place. Sasha, Jose, Peter, and Morris all counted the eggs and got the correct count. When these seven people added their counts together, the sum was 3162. How many eggs were in the pile?

11. What is the remainder when $3^0 + 3^1 + 3^2 + \cdots + 3^{2014}$ is divided by 8?

12. The sum of four two-digit numbers is 221. None of the eight digits is 0 and no two of them are the same. Which of the following is not included among the eight digits? (A) 1 (B) 2 (C) 3 (D) 4 (E) 5

13. Here's a test for divisibility by 7. Double the units digit, lop off the units digit and subtract the double of the units digit from the rest of the number. If the

result is divisible by , then so is the original number. For example, to see that 231 is divisible by 7, compute $23 - 2 = 21$. And 2352 leads to $235 - 4 = 231$ which we just saw, is a multiple of 7. Prove this divisibility test.

14. Chicken McNuggets can be purchased in quantities of 6, 9, and 20 pieces. You can buy exactly 15 pieces by purchasing a 6 and a 9, but you cant buy exactly 10 McNuggets. What is the largest number of McNuggets that can NOT be purchased?

15. How many base-10 three-digit numbers are also three digit numbers in both base-9 and base-11? (AMC 10)

16. Determine the base (-2) representation of 34.

17. ($MA\Theta$) Find the largest integer d for which there are no nonnegative integer solutions (a, b, c) which satisfy the equation $5a + 7b + 11c = d$.

8.3 SET 3

18. On a true-false test of 100 items, every question that is a multiple of 4 is true, and all others are false. If a student marks every item that is a multiple of 3 false and all others true, how many of the 100 items will be correctly answered? (MathCounts)

19. How many whole numbers n, such that $100 \leq n \leq 1000$, have the same number of odd factors as even factors? (MathCounts)

20. Find the base (-4) representation of $33\frac{1}{3}$.

21. (ARML) For a positive integer n, let $C(n)$ be the number of pairs of consecutive 1s in the binary representation of n. For example, $C(183) = C(101101112) = 3$. Compute $C(1) + C(2) + C(3) + \ + C(256)$.

22. The integers 1 through 2010 are written on a white board. The integer 1 is erased. Every integer that is either 7 or 11 greater than an erased integer will be erased. At the end of the process what is the largest integer remaining on the board? (MathCounts)

23. Determine the last two digits of $7^{7^{7^{7}}}$.

24. (IMO) Find the smallest natural number n which has the following properties:
(a) Its decimal representation has 6 as the last digit.
(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

25. To weigh an object by using a balance scale, Brady places the object on one side of the scale and places enough weights on each side to make the two sides of the scale balanced. Bradys set of weights contains the minimum number necessary to measure the whole-number weight of any object from 1 to 40 pounds, inclusive. What is the greatest weight, in pounds, of a weight in Bradys set? (MathCounts)

26. At this stage of 'the course,' we have seen three four-digit numbers which can be permuted by multiplication: $9 \cdot 1089 = 9801, 4 \cdot 2178 = 8712$, and $3 \cdot 2475 = 7425$. What do the three numbers have in common? Can you find other numbers $\underline{abcd}$ with the property that there exists a digit t such that

$t \cdot abcd$ is a four digit number whose digits are a, b, c and d? Prove that any four digit number for which such a t exists must have sum of digits 18.

27. Let S be a subset of $\{1, 2, 3, ..., 50\}$ such that no pair of distinct elements in S has a sum divisible by 7. What is the maximum number of elements in S? (AHSME)

28. Find the least positive integer n for which $\frac{n-13}{5n+6}$ is a non-zero reducible fraction. (AHSME)

29. Let n be the smallest positive integer that is a multiple of 75 and has exactly 75 positive integer divisors, including 1 and itself. Find $n/75$. (AIME)

8.4 SET 4

30. A faulty car odometer proceeds from digit 3 to digit 5, always skipping the digit 4, regardless of position. For example, after traveling one mile the odometer changed from 000039 to 000050. If the odometer now reads 002005, how many miles has the car actually traveled? (AMC 12)

31. Begin with the 200-digit number 9876543210987...43210, which repeats the digits $0 - 9$ in reverse order. From the left, choose every third digit to form a new number. Repeat the same process with the new number. Continue the process repeatedly until the result is a two-digit number. What is the resulting two-digit number? (MathCounts)

32. How many integers from 1 to 1992 inclusive have a base three representation that does not contain the digit 2? (Mandelbrot)

33. Find the remainder when $3^3 \cdot 33^{33} \cdot 333^{333} \cdot 3333^{3333}$ is divided by 100. (Purple Comet)

34. Suppose (a, b, c) is a Pythagorean triple of integers. That is $a^2 + b^2 = c^2$. Prove that 60 is a divisor of the product abc.

35. Determine the sum of all positive three-digit integer numbers that give a reminder 2 when divided by 7, a reminder 4 when divided by 9, and reminder 7 when divided by 12.

36. The number 2^{29} has 9 distinct digits. Which digit is missing?

37. Find the sum of all positive integers n such that $n = d_1^2 + d_2^2 + d_3^2 + d_4^2$ where $d_1 < d_2 < d_3 < d_4$ are the four smallest divisors of n.

8.5 SET 5

38. Let a_n equal $6^n + 8^n$. Determine the remainder upon dividing a_{83} by 49.

39. The numbers 1447, 1005, and 1231 have something in common. Each is a four-digit number beginning with 1 that has exactly two identical digits. How many such numbers are there?

40. One of Euler's conjectures was disproved in the 1960s by three American mathematicians when they showed there was a positive integer such that $133^5 + 110^5 + 84^5 + 27^5 = n^5$. Find the value of n.

41. Let S be a subset of $\{1, 2, 3, \ldots, 1989\}$ such that no two members of S differ by 4 or 7. What is the largest number of elements S can have?

42. What is the smallest positive integer that can be expressed as the sum of nine consecutive integers, the sum of ten consecutive integers, and the sum of eleven consecutive integers?

43. The increasing sequence $3, 15, 24, 48, \ldots$ consists of those positive multiples of 3 that are one less than a perfect square. What is the remainder when the 1994th term of the sequence is divided by 1000?

44. Ninety-four bricks, each measuring $4'' \times 10'' \times 19''$, are to stacked one on top of another to form a tower 94 bricks tall. Each brick can be oriented so it contributes $4''$ or $10''$ or $19''$ to the total height of the tower. How many different tower heights can be achieved using all ninety-four of the bricks?

45. If the number of peanuts in a carton is divided by $15, 16, 17, 18,$ and 19, there is one left over each time. What is the smallest number of peanuts, greater than one, that could be in the carton.

46. A pair of positive integers (x, y) satisfies the equation $31x+29y = 1125$. What is $x + y$?

47. The four consecutive digits a, b, c and d are used to form the four-digit numbers $\underline{abcd}$ and $\underline{dcba}$. What is the greatest common divisor of all numbers of the form $\underline{abcd} + \underline{dcba}$?

48. Exactly one ordered pair of positive integers (x, y) satisfies the equation $37x + 73y = 2016$. What is the sum $x + y$?

8.6 SET 6

49. The integers 1 through 2010 are written on a white board. The integer 1 is erased. Every integer that is either 7 or 11 greater than an erased integer will be erased. At the end of the process what is the largest integer remaining on the board?

50. The 9-digit number $\underline{abbababa3}$ is a multiple of 99 for some pair of digits a and b such that $b > a$. What is ba?

51. Find the sum of all positive integers less than 2006 which are both multiples of 6 and one more than a multiple of 7.

52. Let k be the product of every third positive integer from 2 to 2006. That is, let $k = 2 \cdot 5 \cdot 8 \cdots 2006$. Find the number of zeros at the right end of the decimal representation for k.

53. Consider all ordered pairs (m, n) of positive integers satisfying $59m - 68n = mn$. Find the sum of all the possible values of n in these ordered pairs.

54. The functions $f(x)$ and $g(x)$ are linear functions such that for all x, $f(g(x)) = g(f(x)) = x$. If $f(0) = 4$ and $g(5) = 17$ compute $f(2006)$.