1 Introduction

The *pigeonhole principle*, sometimes called the box principle is a simple but important tool that can be used to solve combinatorial problems. The basic idea is straightforward. Suppose you have $n+1$ pigeons but only n holes. I am grateful to Carl Yerger of Davidson College who prepared the first part of this essay.

1. What can you say about the hole with the largest number of pigeons?

Solution: The hole with the largest number of pigeons has at least two pigeons, because, if you place one pigeon in each of the holes, there's one left over.

What about for a more complicated situation?

2. How many people do you need in a group to ensure that three (or n) people have the same birth month?

Solution: Again, by the pigeonhole principle, the answer is $12(n-1)+1$.

Theorem 1.1 *Let $a_1, a_2, \ldots, a_n$ be not necessarily distinct integers. Then there always exists a subset of these numbers with sum divisible by n.*

Proof: Consider the n integers $s_1 = a_1$, $s_2 = a_1 + a_2$, $s_3 = a_1 + a_2 + a_3$, and so forth up to $a_n = a_1 + a_2 + \cdots + a_n$. If any of these sums (the a_i's) are divisible by n we're finished. (Why?) Thus we can assume that each of the n terms of the form a_i are not divisible by n. How can we apply the pigeonhole principle to finish the proof? **Solution:** Since there are then only $n-1$ remainders, two of the sums, say s_p and s_q where $p < q$ are equal modulo n (that is, have the same remainder). So the difference $s_q - s_p$ is divisible by n.

3. Here's a "friendly" combinatorial question: Suppose n people meet in a room. Everyone shakes hands with some number of people (after all, not everyone in the room may be friends!) After people are finished shaking hands, can we guarantee that there are always two people who have shaken the same number of hands? Draw a diagram that describes the situation if we had a (small) party of four people. Does the guarantee work? Can you generalize your ideas to the general case? **Solution:** Notice that there can't be a person who shakes no hands and a person who shakes all hands. As a result, the possibilities of number of handshakes are either between 0 and $n-2$ or 1 and $n-1$. In either case there are $n-1$ possibilities, but n people, so we can again apply the pigeonhole principle here so that two people are guaranteed to share the same number of handshakes.

4. Here's a problem where the pigeonhole principle may not appear to be useful. Color each point in the xy-plane having integer coordinates either red or blue. Does there exist a rectangle where all the corners are colored the same? Try the problem

on a 2 by 3 integer lattice (so six total points). Can you always color the vertices so that a a rectangle where all the corners are colored the same is forced? How about a 3 by 4 rectangle? What if we extend the length of this rectangle even further to something longer? Can we do this indefinitely, or do we get stuck somewhere? Consider the following key observation. If there are two columns (each of length 3) that have the same coloring, then this always produces a rectangle with all vertices the same color. Why is this true?

Solution: Notice by the pigeonhole principle (again!) that there in each column, at least two points must have the same color. Those points will form the monochromatic rectangle. With this observation in hand, we can obtain a more general result. In particular, notice that there are only 8 ways to color three points either red or blue. Thus, by the pigeonhole principle, if there are nine sets of three points, at least two of them must have the same sequence of colors. But then this pair of sets produces a rectangle where all the corners are colored the same. This completes the proof! This is interesting because we used the pigeonhole principle in a special case of our problem, which then allowed us to solve the entire problem.

5. Here's a geometric question: A target has the form of an equilateral triangle of side length 2. Suppose the target is hit five times and there are now five tiny holes in the target. Must there be two holes that are within distance 1 of each other? **Solution:** Break the triangle into four smaller equilateral triangles, each of side length 1. (In the form of a "triforce") In each of these smaller areas, any two points are distance one from each other. Thus, by the pigeonhole principle, since there are five shots, two shots must go through one of these triangles, and hence they are at least distance one from each other.

6. What if the target is hit 17 times? Now what is the minimal distance of two holes at most? **Solution:** In this case, the minimal distance would be $1/2$ because we can divide each of the triangles of side length 1 into four triangles each of side length $1/2$.

7. Here's a more complicated geometric problem: Prove that two of any six points placed in a 3 by 4 rectangle will have distance less than or equal to $\sqrt{5}$. Take a couple of minutes and try to figure out a way to use the pigeonhole principle. As a hint, think about the types of geometric objects where $\sqrt{5}$ can be used as a length. **Solution:** Take the 3 by 4 rectangle and we will divide it using three "houses" whose base is a 1 by 2 rectangle and whose roof is an isosceles triangle with base length 2 and height 1. We will also use two half-houses, and this pieces together to cover the rectangle. For more details, see Arthur Engel's Problem Solving Strategies, Chapter 4, Problem 33.

8. Another application of the pigeonhole principle is in number theory. We can prove that one of $n+1$ numbers from $\{1, 2, \ldots, 2n\}$ is divisible by another. Consider the following (false) proof: We can think only in terms of divisibility by 2. There are n pairs of numbers where one is a multiple of 2 of the other, for a total of $2n$ numbers. Since we are choosing $n+1$ numbers out of these pairs, by the pigeonhole principle, we will pick both numbers of one of these pairs. What's wrong with this proof? Can you think of an alternative proof? **Solution:** Some of the pairs overlap, and some numbers aren't in any pair at all, so this is a problem. Instead, choose an arbitrary set of $n+1$ numbers and denote them $a_1, a_2, \ldots, a_{n+1}$. Write them in the form $a_i = 2^k b_i$, with b_i odd. Then there are $n+1$ odd numbers $b_1, \ldots b_{n+1}$ from the interval $[1, 2n-1]$. But there are only n odd numbers in this interval. So two of them p, q are such that $b_p = b_q$, and so then one of the numbers, say a_p, is divisible by the other, say a_q.

2 Problems

1. A standard deck of 52 cards has four *suits*, clubs(♣), diamonds(♢), hearts(♡) and spades(♠). Each suit has 13 *values* including ace, two, three, ..., ten, and three face cards, Jack, Queen and King. For each event listed below, find the number of cards that must be selected from a deck to guarantee that the event must happen.

   (a) The set contains a pair; that is, two cards of the same value.

   (b) The set contains three-of-a-kind; that is, three cards of the same value.

   (c) The set contains two pairs (or four of a kind, which is, in a sense, two pair).

   (d) The set contains five spades.

   (e) The set contains two aces.

2. Is it true that from any 30 different natural numbers, not greater than 50 one can choose a pair such that one number in the pair is twice the other.

3. There are 33 students in the class and the sum of their ages is 430 years. Is it true that one can find 20 students in the class such that the sum of their ages greater than 260?

4. The number $7 \cdot 5^{41}$ is a 30 digit number. Prove that some digit appears at least 4 times in the decimal representation.

5. Prove that every set of five vertices of a cube contains three points that are the vertices of an equilateral triangle.

6. Prove that among any set of 52 positive integers, there must exist two whose sum or whose difference is a multiple of 100. Can you make the same claim with just 51 integers?

7. Five points are selected from the square defined by $\{(x, y)| -1 \leq x \leq 1, -1 \leq y \leq 1$. Prove that two of these are within $\sqrt{2}$ of each other.

8. Five lattice points in the plane are given. Show that at least one pair of them has a midpoint which is also a lattice point. A lattice point is a point all of whose coordinates are integers. What is the corresponding problem for lattice points in 3-dimensional space?

9. Prove that 7 has a multiple of the form $111\ldots1$ (called *a repunit*). Is the same proposition true for 19?

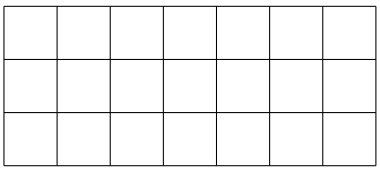
10. Show that every eleven element subset of $S = \{1, 2, 3, 4, \ldots, 62\}$ contains three numbers which are the lengths of the legs of some nondegenerate triangle.

11. Twelve members of a club agree to send cards to one another subject to the following conditions: each person will randomly select six others to send cards to. (Hint: there are only 66 pairs of persons in the club.)

    i. Prove that some pair of persons exchange cards.

    ii. Does the same conclusion hold if each member of the club agrees to send five cards.

12. If each square of a 3-by-7 chessboard is colored either black or white, then the board must contain a rectangle consisting of at least four squares whose corner squares are either all white or all black.



    Show that if the grid is only 3-by-6, there are colorings for which the conclusion fails.

13. Fifty-one points are scattered inside a square with a side of one meter. Prove that some subset of three of these points can be covered by a square with side 20 centimeters.

14. Fifteen children together gathered 100 nuts. Prove that some pair of children gathered the same number of nuts.

15. The integers from 1 to 10 are randomly distributed around a circle. Prove that there must be three neighbors whose sum is at least 17. What about 18? What about 19?

16. The digits $1, 2, 3, \cdots, 9$ are divided up into three groups. Prove that the product of the numbers in one of the groups must exceed 71.

17. A set of 1998 different positive integers is given. None of these numbers can be represented as a sum of two others from the set. What is the least possible value for the largest number from such a set?

18. (2006 Set 1, USAMTS) Suppose each point of the plane is colored red, white, or blue. Prove that there is a rectangle all of whose vertices are the same color (monochromatic).

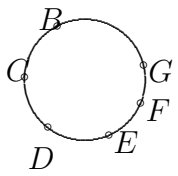
19. Let $P_1, P_2, \ldots, P_8$ be eight points in the set $S = \{(x, y) |\ 0 \leq x \leq 6,\ 0 \leq y \leq 5\}$. Prove that two of the points are within $\sqrt{10}$ of one another.

20. A drawer contains 5 pairs of white gloves, 7 pairs of black gloves, and 4 pairs of gray gloves. The left and right hand gloves are not interchangeable as would be the case with sox. What is the minimum number of gloves that must be selected to be sure that there is a matching pair among them? What if they were sox.

21. The positive integers 1 through 99 are divided into k disjoint subsets so that no two integers whose sum is a multiple of 10 are in the same subset. For example 13 and 37 cannot be in the same subset. What is the smallest possible value of k?

22. A track has n arbitrarily spaced fuel depots. Each depot contains a quantity of gasoline; the total amount of gas is exactly enough to take us around the track once. Prove that, no matter how the gas is distributed, there will be a depot at which an empty car can fill up, proceed around the track picking

up gas at each depot, and complete a full round trip back to its starting depot.



23. Balls of the same size and weight are placed in a container. There are 8 different colors and 90 balls in each color. What is the minimum number of balls that must be drawn from the container in order to get balls of 4 different colors with at least 9 balls for each color?

24. The numbers $1, 4, 9, 16, \ldots, 10000$ are distributed in a 10×10 array of squares. Prove that there must be a horizontal or vertical pair which differ by at least 556.

25. Next suppose the number 1 to 1000 are distributed in a $10 \times 10 \times 10$ array of cells. Prove that some pair of adjacent (sharing a face) differ by at least 37.

26. A subset B of the set of integers from 1 to 100, inclusive, has the property that no two elements of B sum to 125. What is the maximum possible number of elements in B?

27. Prove that for any convex polyhedron, there must be two faces with the same number of edges.

28. A collection of 30 points in the plane with no three collinear is given. Seven lines are drawn. Prove that there exists two points among the 30 for which the line segment connecting them does not hit any of the seven lines.

29. You and the house both pick 10 numbers between 1 and 100. The casino picks randomly you can choose your numbers. Your ticket wins if the casino picks

none of your numbers. Whats the minimum number of tickets that you can buy that will guarantee that at least one of them will be a winner?

3 Final Thoughts and Additional Questions

Here are a couple of questions for further study:
1. Consider the Fibonacci sequence. Prove that for any n there is a Fibonacci number ending with n zeroes.
2. Suppose we have a sequence of n^2+1 integers. Prove that there exists a monotonic subsequence of length $n+1$. A sequence is monotonic if it is either nondecreasing or nonincreasing.

4 References

- Arthur Engel, "Problem Solving Strategies", Springer Problem Books in Mathematics, 1997.
- Bruce Landman, Aaron Robertson, "Ramsey Theory on the Integers," American Mathematical Society Student Mathematical Library, Volume 24.