This essay is about the use of *symmetry* in solving algebra and geometry problems. From Wikipedia we learn that symmetry (from Greek symmetra "measure together") generally conveys two primary meanings. The first is an imprecise sense of harmonious or aesthetically pleasing proportionality and balance; such that it reflects beauty or perfection. The second meaning is a precise and well-defined concept of balance or "patterned self-similarity" that can be demonstrated or proved according to the rules of a formal system: by geometry, through physics or otherwise.[1]

Many of the problems below have the property that the solution is the same even if the names of the variables are permuted. This enables the solver to add or to multiply all the equations together. For example, consider the following problem from the national MATHCOUNTS competition, May 2012.

Example 1 Suppose x, y, and z satisfy $xy = 1, yz = 2$, and $xz = 3$. What is the value of $x^2 + y^2 + z^2$? Note that interchanging the values of the three unknowns does not change the value of $x^2 + y^2 + z^2$. So it makes sense to try putting all the information together, even at the risk of losing some information. The product $xy \cdot yz \cdot xz = 1 \cdot 2 \cdot 3 = 6 = x^2y^2z^2$, so we can solve for x^2, y^2, and z^2 as follows: $x^2 = x^2y^2z^2/y^2z^2 = 6/2^2 = 3/2$, $y^2 = x^2y^2z^2/x^2z^2 = 6/3^2 = 2/3$, and finally $z^2 = x^2y^2z^2/x^2y^2 = 6/1 = 6$. So it follows that $x^2 + y^2 + z^2 = 49/6$.

Example 2 Suppose a pair of real numbers has a sum of 7 and a product of 8. What is the sum of their reciprocals? First, name the numbers, say a and b. Then $1/a + 1/b = (a + b)/ab = 7/8$. Easy if you don't try to find the a and b.

Example 3 Solve the equation $\lfloor \frac{x}{2} \rfloor + \lfloor \frac{x}{3} \rfloor + \lfloor \frac{x}{4} \rfloor + \lfloor \frac{x}{5} \rfloor = 12$. Notice that the function $f(x) = \lfloor \frac{x}{2} \rfloor + \lfloor \frac{x}{3} \rfloor + \lfloor \frac{x}{4} \rfloor + \lfloor \frac{x}{5} \rfloor$ is non-decreasing, and $f(9) = 4+3+2+1 = 10$, while $f(10) = 5 + 3 + 2 + 2 = 12$. If $x < 10$, then $f(x) < 12$ because both $x/2$ and $x/5$ become integers at $x = 10$. The function f is constant on the interval $[10, 12)$ because none of the fractions $x/2, x/3$, etc have integer values on $(10, 12)$.

Example 4 Solve simultaneously:

$$\begin{aligned} x + \lfloor y \rfloor + \langle z \rangle &= 1.1 \\ \langle x \rangle + y + \lfloor z \rfloor &= 2.2 \\ \lfloor x \rfloor + \langle y \rangle + z &= 3.3 \end{aligned}$$

Add the three equations to get $2(x + y + z) = 6.6$. Then subtract the first equation from $x + y + z = 3.3$ to get $y - \lfloor y \rfloor + z - \langle z \rangle = \langle y \rangle + \lfloor z \rfloor = 2.2$. But $\lfloor z \rfloor$ must be the integer part of $\langle y \rangle + \lfloor z \rfloor$, so $\langle y \rangle = 0.2$ and $\lfloor z \rfloor = 2$. The other values $x = 1.0$, $y = 0.2$ and $z = 2.1$ follow from this.

1 Problems

Some of these problems come from the Quantum magazine article by Mark Saul and Titu Andreescu that appeared about 1990.

1. Given that $a+b+c=5$ and $ab+bc+ac=5$, what is the value of $a^2+b^2+c^2$?

2. Solve simultaneously:

$$\begin{aligned} x+2y+z &= 14 \\ 2x+y+z &= 12 \\ x+y+2z &= 18 \end{aligned}$$

3. Solve simultaneously:

$$\begin{aligned} x+y &= 7 \\ x+z &= -2 \\ y+z &= 9 \end{aligned}$$

4. Solve simultaneously:

$$\begin{aligned} xy &= -6 \\ yz &= -2 \\ xz &= 12 \end{aligned}$$

5. Solve simultaneously:

$$\begin{aligned} (x+1)(y+1) &= 24 \\ (y+1)(z+1) &= 30 \\ (x+1)(z+1) &= 20 \end{aligned}$$

6. Solve simultaneously:

$$\begin{aligned} xy-x-y &= 11 \\ yz-y-z &= 14 \\ xz-x-z &= 19 \end{aligned}$$

7. Solve simultaneously:

$$\begin{aligned} x(x+y+z) &= 4 \\ y(x+y+z) &= 6 \\ z(x+y+z) &= 54 \end{aligned}$$

8. Given that a is a real number, solve simultaneously:

$$\begin{aligned} x^2 - xy &= a \\ y^2 - xy &= a(a-1). \end{aligned}$$

9. Solve simultaneously:

$$\begin{aligned} x_2 + x_3 + x_4 + \ldots + x_n &= 1 \\ x_1 + x_3 + x_4 + \ldots + x_n &= 2 \\ x_1 + x_2 + x_4 + \ldots + x_n &= 3 \\ &\vdots \\ x_1 + x_2 + x_3 + \ldots + x_{n-1} &= n. \end{aligned}$$

10. A triangle has sides of lengths $13, 14$, and 15. Its inscribed circle divides each side into two segments, making six altogether. Find the length of each segment.

11. Solve simultaneously:

$$\begin{aligned} xy + xz &= 13 \\ xz + yz &= 25 \\ xy + yz &= 20. \end{aligned}$$

12. Solve simultaneously:

$$\begin{aligned}
2x_1 + x_2 + x_3 + x_4 + x_5 &= 6 \\
x_1 + 2x_2 + x_3 + x_4 + x_5 &= 12 \\
x_1 + x_2 + 2x_3 + x_4 + x_5 &= 24 \\
x_1 + x_2 + x_3 + 2x_4 + x_5 &= 48 \\
x_1 + x_2 + x_3 + x_4 + 2x_5 &= 96
\end{aligned}$$

13. The numbers x, y, z, and w satisfy the system:

$$\begin{aligned}
xyz + yzw + zwx &= 678 \\
xyz + yzw + xyw &= 748 \\
xyz + zwx + xyw &= 700 \\
yzw + zwx + xyw &= 370
\end{aligned}$$

Find the value of $xyzw$.

14. Suppose x, y, and z are positive numbers satisfying

$$x + 1/y = 4, \quad y + 1/z = 1, \quad \text{and} \quad z + 1/x = 7/3.$$

Find the product xyz.

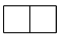
15. Solve the system of equations:

$$\frac{xyz}{x+y} = 7.2, \qquad \frac{xyz}{y+z} = 4, \qquad \frac{xyz}{x+z} = 4.5.$$

16. Solve the equation

$$\frac{x-3}{2001} + \frac{x-5}{1999} + \frac{x-7}{1997} + \frac{x-9}{1995} = \frac{x-2000}{4} + \frac{x-1998}{6} + \frac{x-1996}{8} + \frac{x-1994}{10}.$$

17. Four distinct integers a, b, c and d have the property that when they are added in pairs, the sums $16, 19, 20, 21, 22$ and 25 are obtained. What is the difference between the largest and the smallest.

18. A *grid graph* is a set of vertices in the plane together with the unit edges joining them. For example, the 2×3 grid graph has six vertices and 7 edges. It looks like 

Find the number of edges of the $m \times n$ grid graph.

2 Advanced problems.

19. ABC is a right triangle with area 11 and right angle at C. Suppose $AB = 7$. Find the lengths of the edges AC and BC.

20. ABC is an isosceles triangle with base $BC = 1$ and base angles of $72°$. The angle bisector at B hits the opposite side at point D. Find CD.

21. Suppose that for all positive x,

$$11f(x+1) + 5f(1+1/x) = \log_{10} x.$$

Find $f(6) + f(33) + f(626)$.

22. Find a function f satisfying $x^2 f(x) + f(1-x) = 2x - x^4$ for all real numbers x.

23. Solve the system:

$$x + xy + xyz = 12$$
$$y + yz + yzx = 21$$
$$z + zx + zxy = 30.$$

24. Solve $z^4 + z^3 + z^2 + z + 1 = 0$ over the complex numbers.

25. Suppose all six faces of an $n \times n \times n$ are painted red. Then one of the n^3 unit cubes is randomly selected and tossed like a die. What is the probability that the face obtained is painted? Of course, your answer depends on n. Try this for $n = 1, 2$, and 3. Then make a conjecture and prove your conjecture.

26. Elizabeth is thinking of three positive integers. When she multiplies two of her numbers together and subtracts the third, she obtains the results 4, 172, and 283. What are the three integers?

27. Let a, b and c be real numbers satisfying $a^2 + b^2 + c^2 = 989$ and $(a+b)^2 + (b+c)^2 + (c+a)^2 = 2013$. Find $a + b + c$.

28. Suppose a is a real number for which $a^2 + \frac{1}{a^2} = 14$. What is the largest possible value of $a^3 + \frac{1}{a^3}$?

29. If a, b and c are three distinct numbers such that

$$a^2 - bc = 7, \qquad b^2 + ac = 7, \quad \text{and} \quad c^2 + ab = 7,$$

then what is the value of $a^2 + b^2 + c^2$?

30. How many pairs of positive integers (a, b) are there such that a and b have no common factors greater than 1 and

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer?

31. Suppose a, b, c are integers satisfying $1 \leq a < b < c$ and $a^2 + b^2 + c^2 = 14(a + b + c)$. Find all ordered triplets (a, b, c) satisfying the given conditions. How many such triplets are there?

32. Find the volume of a rectangular box whose left side, front side, and bottom have areas of 10 square inches 15 square inches and 294 square inches, respectively.

33. Four numbers are written in a row. The average of the first two numbers is 5. The average of the middle two numbers is 4 and the average of the last two numbers is 10. What is the average of the first and last numbers?

34. The 8×10 grid below has numbers in half the squares. These numbers indicate the number of mines among the squares that share an edge with the given one. Squares containing numbers do not contain mines. Each square that does not have a number either has a single mine or nothing at all. How many mines are there?

	1		1		2		2		1
1		2		3		2		3	
	3		2		3		2		2
1		3		2		1		3	
	3		2		1		1		2
2		4		1		1		2	
	3		3		2		3		1
1		2		2		3		2	

35. Find the real numbers m, n such that $m + n = 3$, and $m^3 + n^3 = 117$. What is the value of $m^2 + n^2$?

36. For how many integers n is the value of $\frac{n}{50-n}$ the square of an integer?

37. Four positive integers a, b, c and d satisfy $abcd = 10!$. What is the smallest possible sum $a+b+c+d$?

38. Suppose a, b, c, and d are positive integers satisfying

$$ab + cd = 38$$

$$ac + bd = 34$$

$$ad + bc = 43$$

What is $a+b+c+d$?

39. Given the following system of equations

$$\begin{aligned} \frac{1}{x}+\frac{1}{y} &= \frac{1}{3} \\ \frac{1}{x}+\frac{1}{z} &= \frac{1}{5} \\ \frac{1}{y}+\frac{1}{z} &= \frac{1}{7} \end{aligned}$$

What is the value of the fraction $\frac{z}{y}$?

40. The areas of three faces of a rectangular parallelepiped are 18, 40, and 80. Find its volume.

41. Let x, y be positive integers with $x > y$. If $1/(x+y) + 1/(x-y) = 1/3$, find $x^2 + y^2$.

42. If x, y, and z are real numbers that satisfy

$$\frac{x}{y-6} = \frac{y}{z-8} = \frac{z}{x-10} = 3.$$

What is the value of $x+y+z =$

43. If $ac + ad + bc + bd = 68$ and $c + d = 4$, what is the value of $a+b+c+d$?

44. Find the number of ordered quadruples of positive integers (a, b, c, d) such that a, b, c, d are all (not necessarily distinct) factors of 30 and $abcd > 900$.

45. Let x, y, z be positive real numbers satisfying the simultaneous equations

$$x(y^2 + yz + z^2) = 3y + 10z$$
$$y(z^2 + zx + x^2) = 21z + 24x$$
$$z(x^2 + xy + y^2) = 7x + 28y.$$

Find $xy + yz + zx$.

46. Let $f : \mathtt{R} \to \mathtt{R}$ be a function which satisfies $f(x + 31) = f(31 - x)$ for all real numbers x. If f has exactly three real roots, what is the sum of the roots.

47. What is $i + 2i^2 + 3i^3 + 4i^4 + \cdots + 60i^{60}$? Recall that $i = \sqrt{-1}$.

48. Three points are chosen independently and at random on a circle (using the uniform distribution). What is the probability that the center of the circle lies inside the resulting triangle?

49. What is the largest 2-digit prime factor of the integer $\binom{200}{100}$?

50. How many of the first 1000 positive integers can be expressed in the form

$\lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor$,

where x is a real number, and $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z?

51. Evaluate the product $(\sqrt{5}+\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}-\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7})$.

52. What is the largest possible distance between two points, one on the sphere of radius 19 with center $(-2, -10, 5)$ and the other on the sphere of radius 87 with center $(12, 8, -16)$?

53. Suppose D is a set of digits, and S is the set of all n-digit numbers that can be built from the digits of D. To what extent is a member of S determined by the sum of the other members of S.

(a) Let N be a 3-digit number with three different digits. The set S of 3-digit numbers made from the digits of N has six members. Suppose five of these have the sum 2003. What is the sixth number?

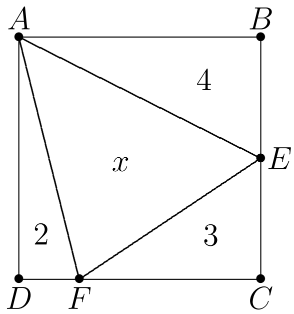
(b) Let N be a four digit number with three distinct digits, and suppose the sum of all the other numbers, excluding N is 39529. What is N?

(c) Now suppose $N = \underline{abcd}$ is a four digit number with four different digits. What is the sum of all 24 members of the set S? Suppose the sum of the digits of N is 14. What are the possible values of N? Build a problem like the two above by leaving out one of the four digit numbers from the sum.

54. Find the area of the region enclosed by the graph of $|x - 60| + |y| = \left|\frac{x}{4}\right|$.

55. Without a calculator, compute $\sqrt{(31)(30)(29)(28) + 1}$.

56. A square $ABCD$ of side s is shown below. E belongs to $\overline{BC}$ and F belongs to $\overline{CD}$. Triangles ABE, ECF, and ADF have areas $4, 3$ and 2, respectively. What is the area of triangle AEF?



57. Can you find a problem like the one above for which there are two answers? IE, is there a set of area values so that there are two x values and two s values?

58. Four points are chosen independently and at random on the surface of a sphere (using the uniform distribution). What is the probability that the center of the sphere lies inside the resulting tetrahedron?

59. Find $x^2 + y^2$ if x and y are positive integers such that $xy + x + y = 71$ and $x^2y + xy^2 = 880$.

60. Find all ordered triples of numbers (x, y, z) that satisfy

$$xy + x + y = 11,\ xz + x + z = 17,\ yz + y + z = 23.$$

61. The numbers $1, 2, 3, \ldots, 10$ are written on a blackboard. You are allowed to erase any two numbers a and b and replace them with $a + b - 1$. After nine of these operations, what number is left?

62. The numbers $1, 2, 3, \ldots, 100$ are written on a blackboard. You are allowed to erase any two numbers a and b and replace them with $ab + a + b$. After 99 of these operations, what number is left?

63. Let a, b, and c be real numbers such that $2a-7b+9c = 6$ and $9a+6b-2c = 7$. Find $a^2 - b^2 + c^2$.

64. Let x and y be real numbers with $x+y = 1$ and $(x^2+y^2)(x^3+y^3) = 12$. What is the value of $x^2 + y^2$?

65. How many triples (a, b, c) of real numbers satisfy the equations $ab = c, ac = b$ and $bc = a$?

66. Given that a and b are positive real numbers with $a + b = 4$, what is the minimum value of

$$\left(1+\frac{1}{a}\right)\left(1+\frac{1}{b}\right)?$$

Let x, y and z be real numbers satisfying both $x+y+z = 10$ and $x^2+y^2+z^2 = 50$. What is the value of $(x + 2y + 3z)^2 + (2x + 3y + z)^2 + (3x + y + 2z)^2$?

67. Four numbers $a \le b \le c \le d$ are added in pairs to get the six values $4, 6, 8, 18, 20, 22$. What is $a + b + c + d$?

68. Four numbers $a \le b \le c \le d$ are added in pairs. Five of the sums are values $8, 10, 18, 20, 22$. What is $a + b + c + d$?

69. How many real numbers satisfy the equation $\dfrac{x(x^2 - 1)(x^2 - 4)}{x - |x|} = 0$?

70. Let $f(x) = x^2 - bx + c$ be a real function such that $f(5-x) = f(5+x)$ for all real numbers x. What is the minimum value of the product $f(1)f(2)$?

71. What is
$$\frac{1+3+5\cdots+2019}{2+4+6\cdots+2018}?$$

72. Erick makes a list of the integers 0 to 999, inclusive, that do not contain the digit 7. What is the median entry on Erick's list?

73. In a rectangular box, the distances from vertex A to the three vertices with which it shares a face, but not an edge are 5 cm, 6 cm, and 7 cm. What is the volume of the box? Express your answer in simplest radical form.

74. Positive integers x and y satisfy the equation
$$x^3 - y^3 = 218.$$
Find $x + y$.

75. Find positive integers x and y satisfying
$$x^3 + y^3 = 341.$$
item How many positive integers less that or equal to 2018 have strictly more 1's than 0's in their binary expansion?

76. Determine $w^2 + x^2 + y^2 + z^2$ if
$$\frac{x^2}{2^2-1}+\frac{y^2}{2^2-3^2}+\frac{z^2}{2^2-5^2}+\frac{w^2}{2^2-7^2}=1$$
$$\frac{x^2}{4^2-1}+\frac{y^2}{4^2-3^2}+\frac{z^2}{4^2-5^2}+\frac{w^2}{4^2-7^2}=1$$
$$\frac{x^2}{6^2-1}+\frac{y^2}{6^2-3^2}+\frac{z^2}{6^2-5^2}+\frac{w^2}{6^2-7^2}=1$$
$$\frac{x^2}{8^2-1}+\frac{y^2}{8^2-3^2}+\frac{z^2}{8^2-5^2}+\frac{w^2}{8^2-7^2}=1$$

77. Let $\underline{uvw}$ be a three-digit number. Find digits a, b, c, not necessarily distinct, for which
$$\underline{abc} + \underline{cab} - \underline{bca} = \underline{uvw}.$$

(a) $\underline{uvw} = 608$

(b) $\underline{uvw} = 708$

(c) $\underline{uvw} = 707$

78. Solve the following system of equations for real numbers w, x, y and z:

$$1w + 8x + 3y + 5z = 20$$

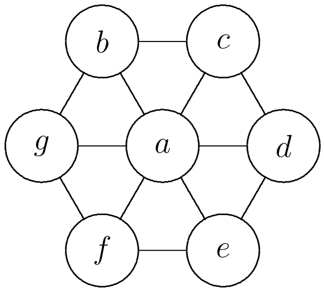
$$4w + 7x + 2y + 3z = -20$$

$$6w + 3x + 8y + 7z = 20$$

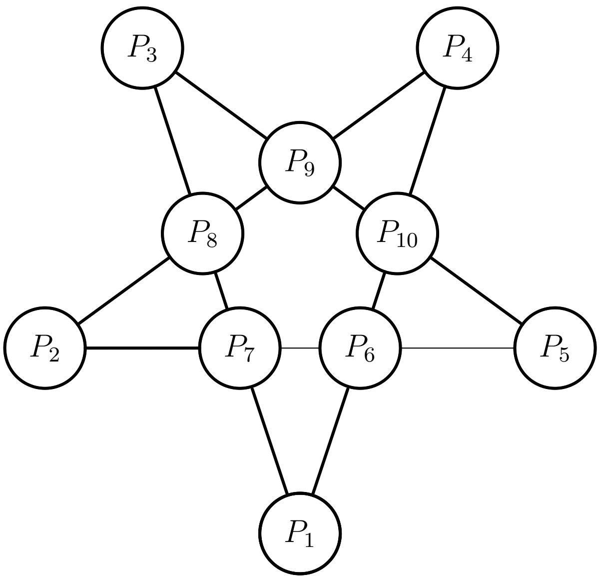
$$7w + 2x + 7y + 3z = -20$$

3 Projects

79. **Highly Unmagic Hexagram.** I'm grateful to Sam Vandervelde for this problem. Distribute the numbers 1 through 7 in the seven circles to that the nine lines $bc, cd, de, ef, fg, gb, abe, acf$ and adg all have **different** line sums.



80. Arrange the numbers $1, 2, 3, 4, 5, 6, 8, 9, 10, 12$ in the ten locations so that the sum of the four numbers along each of the five lines is the same.



81. Now try the same problem with the numbers $1, 5, 7, 11, 18, 21, 24, 33, 42, 43$.