28,800 Extremely Magic 5×5 Squares

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1 Abstract

Using an algorithm from the paper "Creating Semi-Magic, Magic and Extra Magic $n \times n$ Squares when n is Odd", we define 28,800 extra magic (or panmagic) 5×5 squares. We show that these 5×5 extra magic squares are also extremely magic. This means that all of these 28,800 extra magic 5×5 squares have the same 120 magic 5 element subsets whose sum is 65. Of course, these subsets include the 5 rows and 5 columns and the 10 generalized diagonals. We also discuss additional magic properties. We also define 28,800 similarity mappings on these 5×5 extremely magic squares that map these 120 magic 5 element subsets onto each other. As always, these 28,800 similarity mappings form a group. If we use any one of the 5×5 extremely magic squares as the domain and use any other as the range then the function that we create will be one of these 28,800 similarity mappings. Thus, we can loosely say that the 28,800 magic squares themselves form a group. However, the graph of a group is a more accurate term.

At the end, we state a variation of Hall's Marriage Theorem that can greatly enhance this paper. Much of this paper can be generalized, but the 5×5 magic squares have properties that are unique.

2 Introduction

Using an algorithm developed in the paper "Creating Semi-Magic, Magic and Extra Magic $n \times n$ Squares when n is Odd", we define the following 5×5 Matrix. All the results of this paper are derived from this matrix. Throughout this paper, we use the term *line* to mean any of the 20 rows, columns, and generalized diagonals.

(a_0, A_0)	(a_1, A_1)	(a_2, A_2)	(a_3, A_3)	(a_4, A_4)
(a_3, A_2)	(a_4, A_3)	(a_0, A_4)	(a_1, A_0)	(a_2, A_1)
(a_1, A_4)	(a_2, A_0)	(a_3, A_1)	(a_4, A_2)	(a_0, A_3)
(a_4, A_1)	(a_0, A_2)	(a_1, A_3)	(a_2, A_4)	(a_3, A_0)
(a_2, A_3)	(a_3, A_4)	(a_4, A_0)	(a_0, A_1)	(a_1, A_2)

Fig. 1, A 5×5 Matrix

Note that this matrix contains all the ordered pairs $(a_i, A_j), i = 0, 1, 2, \ldots, 4, j = 0, 1, 2, \ldots, 4$. We can also define the dual 5×5 matrix where we change each (a_i, A_j) in Fig. 1 to (a_j, A_i). Thus, the second row of the dual matrix would read $(a_2, A_3), (a_3, A_4), (a_4, A_0), (a_0, A_1), (a_1, A_2)$. In Fig. 1 we agree that a_0, a_1, a_2, a_3, a_4 is any arbitrary but fixed permutation of the integers $0, 1, 2, 3, 4$. Also, A_0, A_1, A_2, A_3, A_4 is any arbitrary but fixed permutation of $0, 1, 2, 3, 4$.

In Fig. 1, if we permute a_0, a_1, a_2, a_3, a_4 in all possible ways and permute A_0, A_1, A_2, A_3, A_4 in all possible ways we can define $5! \cdot 5! = 14,400$ different 5×5 matrices.

For each ordered pair (a_i, A_j) in Fig. 1, let us assign the number $(a_i, A_j)^{\#} = a_i + 5(A_j - 1)$ where a_i, A_j are the numerical values that have been assigned to a_i, A_j.

Now $1 \leq a_i \leq (a_i, A_j)^{\#} = a_i + 5(A_j - 1) \leq 5 + 5(5 - 1) = 5^2$. That is, $1 \leq (a_i, A_j)^{\#} \leq 25$.

We now show that $(a_i, A_j)^{\#} = (a_k, A_e)^{\#}$ if and only if $a_i = a_k$ and $A_j = A_e$.

First, suppose $A_j \neq A_e$ and by symmetry suppose $A_j < A_e$. We show that $(a_i, A_j)^{\#} < (a_k, A_e)^{\#}$. That is, we show $(a_i, A_j)^{\#} = a_i + 5(A_j - 1) < a_k + 5(A_e - 1) = (a_k, A_e)^{\#}$. This is equivalent to $a_i - a_k < 5(A_e - A_j)$.

Since $a_i, A_j, a_k, A_e \in \{0, 1, 2, 3, 4\}$ and $A_j < A_e$ we see that $a_i - a_k < 5 \leq 5(A_e - A_j)$. Therefore, $a_i - a_k < 5(A_e - A_j)$.

If $A_j = A_e$ then obviously $(a_i, A_j)^{\#} = (a_k, A_e)^{\#}$ if and only if $a_i = a_k$. Therefore, $(a_i, A_j)^{\#} = (a_k, A_e)^{\#}$ if and only if $a_i = a_k$ and $A_j = A_e$.

From this we see that $\left\{(a_i, A_j)^{\#} : a_i, A_j \in \{0, 1, 2, 3, 4\}\right\} = \{1, 2, 3, 4, \cdots, 25\}$ since the matrix contains all the ordered pairs $(a_i, A_j), i = 0, 1, 2, \ldots, 4, j = 0, 1, 2, \ldots, 4$. Thus, we can place $1, 2, 3, \cdots, 25$ in the 5×5 matrix. Let us now observe the following about Fig. 1, namely that each row contains each of a_0, a_1, a_2, a_3, a_4 and contains each of A_0, A_1, A_2, A_3, A_4. Also, each column contains each of a_0, a_1, a_2, a_3, a_4 and contains each of A_0, A_1, A_2, A_3, A_4. Also, each of the two main diagonals and each of the 8 generalized diagonals contains each of a_0, a_1, a_2, a_3, a_4 and contains each of A_0, A_1, A_2, A_3, A_4.

Note that there are 10 generalized diagonals in Fig. 1 when we count the two main diagonals. One example of a generalized diagonal is the set $\{(a_1, A_1), (a_0, A_4), (a_4, A_2), (a_3, A_0), (a_2, A_3)\}$. Another example is $\{(a_1, A_1), (a_3, A_2), (a_0, A_3), (a_2, A_4), (a_4, A_0)\}$. If we use the code $(a_i, A_j)^{\#} = a_i + 5(A_j - 1)$ we now show that the sum of the 5 numbers in each row, in each column and in each generalized diagonal equals to $\frac{1}{5}(1 + 2 + \cdots + 25) = \frac{25}{10}(26) = 65$. This means that the 5×5 Matrix of Fig. 1 becomes an extramagic (or panmagic) 5×5 square.

For example, the sum of the 5 numbers in the 1st main diagonal equals

$$\begin{aligned}
& (a_0, A_0)^{\#} + (a_4, A_3)^{\#} + (a_3, A_1)^{\#} + (a_2, A_4)^{\#} + (a_1, A_2)^{\#} \\
= \; & [a_0 + 5\,(A_0 - 1)] + \cdots + [a_1 + 5\,(A_2 - 1)] \\
= \; & \sum_{i=0}^{4} a_i + 5 \sum_{i=0}^{4} A_i - 5 \cdot 5 \\
= \; & \sum_{i=1}^{5} i + 5 \sum_{i=0}^{4} i - 5 \cdot 5 \\
= \; & 15 + 5 \cdot 15 - 25 \\
= \; & 65,
\end{aligned}$$

since $\{a_0, a_1, a_2, a_3, a_4\} = \{A_0, A_1, A_2, A_3, A_4\} = \{0, 1, 2, 3, 4\}$. This reasoning is the same for all lines.

3 A Specific Example

In Fig. 1, we now let $(a_0, a_1, a_2, a_3, a_4) = (3, 5, 1, 2, 4)$ and $(A_0, A_1, A_2, A_3, A_4) = (2, 1, 5, 4, 3)$. Using the code $(a_i, A_j)^{\#} = a_i + 5\,(A_j - 1)$ we have the extra magic 5×5 square of Fig. 2.

8	5	21	17	14
22	19	13	10	1
15	6	2	24	18
4	23	20	11	7
16	12	9	3	25

Fig. 2 An Extra Magic 5×5 square

The magic 5×5 square of Fig. 2 is actually extremely magic and we show later how to use this extremely magic 5×5 square to generate $28,800$ different, extremely magic 5×5 squares.

Note 1. Suppose that instead of using the integers 1 though 25, we are interested in putting any numbers $a_1, a_2, \ldots, a_{25}$ in the 25 positions of the 5×5 matrix so that the sum of the 5 numbers in each row, column, and generalized diagonal is the same. This leads to a system of linear equations in the variables $a_1, a_2, \ldots, a_{25}$. Professor Ben Klein has shown that the rank of this system of linear equations is 16. Thus, it is possible to find a subset of nine letters $a_{i_1}, a_{i_2}, \ldots, a_{i_9}$, and arbitrarily specify the values of these nine letters. Then we can compute the unique values of the other 16 letters in terms of the 9 given letters. In the 5×5 matrix of Fig. 1, let us now use the code $(a_i, A_j)^{\#} = a_i + A_j$ and then place these values $(a_i, A_j)^{\#}$ in the 5×5 matrix. We note from Fig. 1 that the sum of the 5 numbers on each line is always the same value

$$\sum_{i=0}^{4} a_i + \sum_{i=0}^{4} A_i.$$

Now $(a_i, A_j)^{\#} = a_i + A_j = (a_i + x, A_j - x)^{\#}$. Therefore, if we replace each a_i by $a_i + x$ and replace each A_j by $A_j - x$, we see that $(a_i, A_j)^{\#} = a_i + A_j = (a_i + x, A_j - x)^{\#}$. Thus, there is no loss in generality to always let $a_0 = 0$. Therefore there are nine degree of freedom among the ten letters $a_0, a_1, \ldots, a_4, A_0, A_1, \ldots, A_4$, since $a_0 = 0$. Given that the rank of the 5×5 matrix in Fig. 1 is 16, it is straightforward to show that the 5×5 matrix of Fig. 1 with the code $(a_i, A_j)^{\#} = a_i + A_j$ where $a_0 = 0$ and where $a_1, a_2, a_3, a_4, A_0, \ldots, A_4$ vary over the set of numbers, will give all of the 5×5 matrices with a common sum for all lines. The panmagic square below is one example of the attractive mathematics we can create from this theory. If we add A to each of the 25 squares, we will have 9 independent variables and we can easily find $a_0 = 0, a_1, a_2, \ldots, a_4, A_0, A_1, \ldots, A_4$ that will realize these 9 independent variables.

0	B+I	C+H	D+G	E+F
D+H	E+G	F	B	C+I
B+F	C	D+I	E+H	G
E+I	H	B+G	C+F	D
C+G	D+F	E	I	B+H

is an extra magic 5×5 square, where B, C, D, E, F, G, H, I are arbitrary, and the common sum is $B + C + D + E + F + G + H + I$.

4 The Extreme Magicness of Figs. 1, 2

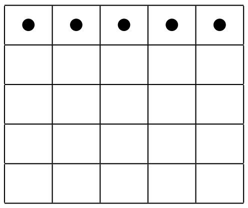
In this section, we derive the extreme magicness of Fig.1 , 2. From Note 1 it will then be self-evident that any panmagic 5×5 square will also be extremely magic. This is stated in [3], but it is not derived in [3]. Also the proof in [3] is far from self-evident.

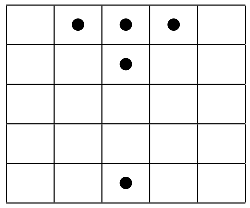
In the rest of this paper, we assume that $(a_0, a_1, a_2, a_3, a_4)$ and $(A_0, A_1, A_2, A_3, A_4)$ have been arbitrarily fixed by Fig. 2. In Fig. 1, let us now permute A_0, A_1, A_2, A_3, A_4 in all possible ways to create 120 different 5 element sets of ordered pairs $\{(a_0, A_{i0}), (a_1, A_{i1}), (a_2, A_{i2}), (a_3, A_{i3}), (a_4, A_{i4})\}$.

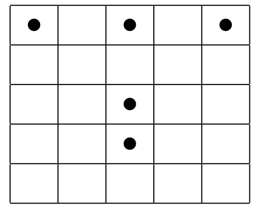
Thus, one example would be the 5 element set $\{(a_0, A_1), (a_1, A_2), (a_2, A_4), (a_3, A_0), (a_4, A_3)\}$. These 120 5-element sets of ordered pairs will define 120 different 5 element subsets of the Fig. 1 matrix. When we use the code $(a_i, A_j)^{\#} = a_i + 5(A_j - 1)$, each of these 120 different 5 element subsets of the Fig. 1 matrix will be a magic 5 element subset of the Fig. 1 matrix since the sum of the 5 elements in the set will be equal to 65. This follows since each of these 120 different 5 element subsets of Fig. 1 will contain each of the letters a_0, a_1, a_2, a_3, a_4 and contain each of the letters A_0, A_1, A_2, A_3, A_4 and since the code is $(a_i, A_j)^{\#} = a_i + 5(A_j - 1)$.

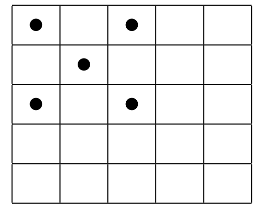
Also, of course, $\{a_0, a_1, a_2, a_3, a_4\} = \{0, 1, 2, 3, 4\}$ and $\{A_0, A_1, A_2, A_3, A_4\} = \{0, 1, 2, 3, 4\}$.

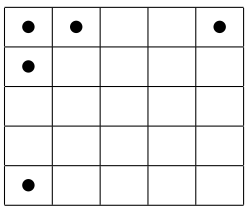
After we go through all of the 120 sets $\{(a_0, A_{i0}), (a_1, A_{i1}), (a_2, A_{i2}), (a_3, A_{i3}), (a_4, A_{i4})\}$ we have the classification of the 120 magic 5 element subsets of Fig. 1 that is shown below. In these drawings, we also state the number of times that each magic 5 element configuration appears in Fig. 1. We note that there are 16 different configurations that are listed below and most of these 16 configurations can appear many times in the 5×5 matrix of Fig. 1. For example, the configuration of drawing G can appear 16 times in the 5×5 matrix of Fig. 1.

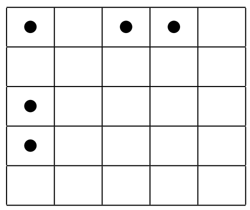

A. 10 (Rows, Columns)

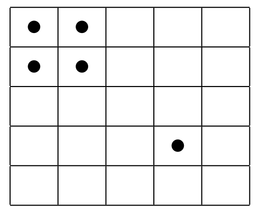

B. 12

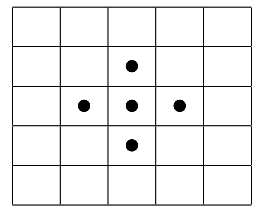

C. 8

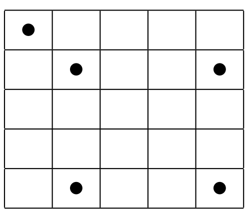

D.9

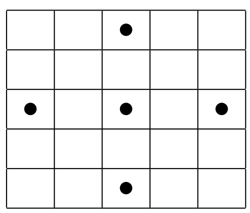

E. 4

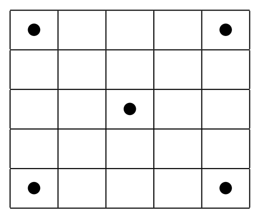

F. 16

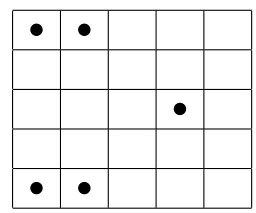

G. 16

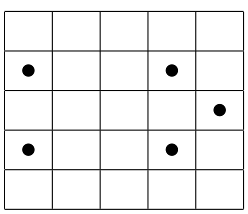

H. 9

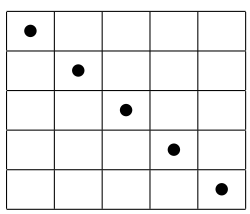

I. 4

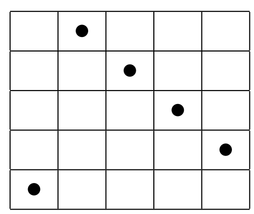

J. 1

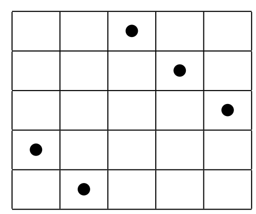

K. 1


L. 8


M. 12


N. 2 (Main Diagonals)


O. 4


P. 4

We note that the sum of the numbers in these 16 drawings equals 120 which means that these configurations appear 120 times in Figs. 1, 2. We also note that every time that one of the 16 configurations appears in Fig. 1 this set of 5 elements will contain all of a_0, a_1, a_2, a_3, a_4 and contain all of A_0, A_1, A_2, A_3, A_4. These 120 subsets of Fig. 1 are also the only 5 elements subsets of Fig. 1 that contain all of a_0, a_1, a_2, a_3, a_4 and contain all of A_0, A_1, A_2, A_3, A_4. Also, these 120 subsets of Fig. 1 are the only five-element subsets of Fig. 1 such that each pair of points in the set lie on a line.

We note that drawing A represents the horizontal rows and vertical columns for a total of 10. Also, drawing N represents the two main diagonals for a total of 2. Also, drawings O, P together represent the 8 generalized diagonals for a total of 8.

Let us now focus our attention on drawing E. This configuration appears 4 times in Figs. 1, 2. These are

$$\{(a_0, A_0), (a_1, A_1), (a_3, A_2), (a_4, A_4), (a_2, A_3)\} = \{8, 5, 22, 14, 16\},$$
$$\{(a_2, A_3), (a_4, A_1), (a_3, A_4), (a_0, A_0), (a_1, A_2)\} = \{16, 4, 12, 8, 25\},$$
$$\{(a_1, A_2), (a_0, A_1), (a_3, A_0), (a_2, A_3), (a_4, A_4)\} = \{25, 3, 7, 16, 14\},$$
$$\{(a_4, A_4), (a_2, A_1), (a_3, A_3), (a_1, A_2), (a_0, A_0)\} = \{14, 1, 17, 25, 8\}.$$

Note that each of these four 5 element subsets contain all of a_0, a_1, a_2, a_3, a_4 and contain all of A_0, A_1, A_2, A_3, A_4 and the sum of the 5 numbers in each of these four 5 element subsets is always 65. The sum of the 5 elements in each of the 120 magic subsets will always equal 65.

In Section 10, we state that a variation of Hall's Marriage Theorem can be used to show that these 5×5 extremely magic squares are actually much more magic than what we have studied in this section. However, we consider Hall's Marriage Theorem to be outside of the scope of this paper. So in Section 10 we state these results without proof. In Section 11, we state a variation of Hall's Marriage Theorem.

Note 2. There is a large number of other properties embedded in the panmagic 5×5 square of Fig. 1 when we use the code $(a_i, A_j)^\# = a_i + 5\,(A_j - 1)$ or $(a_i, A_j)^\# = a_i + A_j$. Consider for example, the two sets of doubleton entries $\{(a_1, A_2), (a_3, A_4)\}$ and $\{(a_1, A_4), (a_3, A_2)\}$. Now $(a_1, A_2)^\# + (a_3, A_4)^\# = (a_1, A_4)^\# + (a_3, A_2)^\# = (a_1 + a_3) + (A_2 + A_4)$. Therefore, in any panmagic 5×5 square, these two doubleton sums must always be equal. Thus, in Fig.2 we see that $12 + 25 = 22 + 15 = 37$. In Fig. 1, consider the 5 element sets $\{(a_1, A_2), (a_0, A_1), (a_2, A_3), (a_1, A_0), (a_3, A_4)\}$ and $\{(a_1, A_4), (a_0, A_2), (a_2, A_1), (a_1, A_3), (a_3, A_0)\}$. It is obvious that the sum of the 5 elements in each of these two sets is the same. Thus, in Fig. 2, we see that $25 + 3 + 16 + 10 + 12 = 15 + 23 + 1 + 20 + 7 = 66$. Reference [3] does not mention these properties.

5 Similarity Mappings on the Fig. 1 Matrix

Remember that a_0, a_1, a_2, a_3, a_4 and A_0, A_1, A_2, A_3, A_4 can be any arbitrary permutations of the integers $0, 1, 2, 3, 4$. In this paper we have agreed that a_0, a_1, a_2, a_3, a_4 and A_0, A_1, A_2, A_3, A_4 have been fixed by Fig. 2.

In the Fig. 1 matrix, we note that a_0, a_1, a_2, a_3, a_4 can be permuted in $5! = 120$ different ways and A_0, A_1, A_2, A_3, A_4 can be permuted in $5! = 120$ different ways. Thus, we can use Fig. 1 to define $120 \cdot 120 = 14,400$ different arrangements (or permutations) of the 25 ordered pairs $\{(a_i, A_j) : i, j \in \{0, 1, 2, 3, 4\}\}$.

Each of these arrangements (or permutations) will define an extremely magic 5×5 square and each of these magic 5×5 squares will have the common 120 magic 5 element subsets that are listed in Section 4.

We observe that the arrangement of the five (a_i, A_j)'s in the top row of Fig. 1 will completely determine the arrangement (or permutation) of all $25\,(a_i, A_j)$'s in Fig. 1. For example, if the top row of Fig. 1 reads $(a_1, A_0), (a_3, A_4), (a_0, A_3), (a_4, A_1), (a_2, A_2)$ we know that we are using the permutations $\begin{bmatrix} a_0, a_1, a_2, a_3, a_4 \\ a_1, a_3, a_0, a_4, a_2 \end{bmatrix}$ and $\begin{bmatrix} A_0, A_1, A_2, A_3, A_4 \\ A_0, A_4, A_3, A_1, A_2 \end{bmatrix}$. These permutations determine all 25 entries in the Fig. 1 matrix and these 25 entries are shown in Fig. 3.

(a_1, A_0)	(a_3, A_4)	(a_0, A_3)	(a_4, A_1)	(a_2, A_2)
(a_4, A_3)	(a_2, A_1)	(a_1, A_2)	(a_3, A_0)	(a_0, A_4)
(a_3, A_2)	(a_0, A_0)	(a_4, A_4)	(a_2, A_3)	(a_1, A_1)
(a_2, A_4)	(a_1, A_3)	(a_3, A_1)	(a_0, A_2)	(a_4, A_0)
(a_0, A_1)	(a_4, A_2)	(a_2, A_0)	(a_1, A_4)	(a_3, A_3)

Fig. 3 A Similarity Mapping of Fig. 1

These $14,400$ permutations form a group of similarity mappings on the 5×5 magic square of Fig. 1 and we explain this in a moment. The group structure of the $14,400$ permutations is $S_5 \times S_5$ where S_5 is the symmetric group on $\{0,1,2,3,4\}$. We go deeper into this in Section 6. We will also soon show why these $14,400$ permutations of Fig. 1 are similarity mappings on the magic squares.

Suppose, x_1, x_2, x_3, x_4, x_5 and $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5$ are any arbitrary magic 5 element subsets of the Fig. 1 matrix whose elements x_1, x_2, x_3, x_4, x_5 and $\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5$ are listed in any arbitrary order. It is easy to see that we can choose permutations $\begin{bmatrix} a_0, & a_1, & a_2, & a_3, & a_4 \\ a_{i0}, & a_{i1}, & a_{i2}, & a_{i3}, & a_{i4} \end{bmatrix}$ and $\begin{bmatrix} A_0, & A_1, & A_2, & A_3, & A_4 \\ A_{j0}, & A_{j1}, & A_{j2}, & A_{j3}, & A_{j4} \end{bmatrix}$ so that the corresponding similarity mapping f will map $f(\{x_1, x_2, x_3, x_4, x_5\}) = \{\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4, \bar{x}_5\}$ with $f(x_i) = \bar{x}_i$ for all $i = 0,1,2,3,4$. We also note that the permutations $\begin{bmatrix} a_0, a_1, a_2, a_3, a_4 \\ a_4, a_2, a_0, a_3, a_1 \end{bmatrix}$ and $\begin{bmatrix} A_0, A_1, A_2, A_3, A_4 \\ A_4, A_1, A_3, A_0, A_2 \end{bmatrix}$ will define a similarity mapping f that rotates the Fig. 1 matrix $90°$ counterclockwise.

We can also show that there is no permutations $\begin{bmatrix} a_0, & a_1, & a_2, & a_3, & a_4 \\ a_{i0}, & a_{i1}, & a_{i2}, & a_{i3}, & a_{i4} \end{bmatrix}$ and $\begin{bmatrix} A_0, & A_1, & A_2, & A_3, & A_4 \\ A_{j0}, & A_{j1}, & A_{j2}, & A_{j3}, & A_{j4} \end{bmatrix}$ that will flip the matrix of Fig. 1 over. The dual matrix of Fig. 1 is needed to flip Fig. 1 over.

In Fig. 4 we have numbered the squares of Fig. 1 $1, 2, 3, \cdots, 25$. Thus, $1 = (a_0, A_0), 2 = (a_1, A_1), 3 = (a_2, A_2)$, etc.

1	2	3	4	5
6	7	8	9	10
11	12	13	14	15
16	17	18	19	20
21	22	23	24	25

Fig. 4 Numbering the 25 squares

Of course, each of the $14,400$ permutations that we have defined will define a permutation on the Fig. 4 matrix. Each permutation will be a similarity mapping on the 5×5 matrix in the sense that it maps the 120 magic 5 element subsets of the matrix onto each other. This fact by itself would imply that each mapping of Figs. 1, 2, 4 would produce an extremely magic 5×5 square from any extremely magic 5×5 square and all such magic squares would have the same 120 magic 5 element subsets. The similarity mappings on any structure always form a group.

From Note 1 it is obvious that any panmagic 5×5 square is also extremely magic. Thus, these 14400 permutations would produce a panmagic 5×5 square from any panmagic 5×5 square. Reference [3] does not mention these similarity mappings.

6 A Basis for the Similarity Mappings

As stated in Section 5, the arrangement of the five (a_i, A_j)'s in the top row of Fig. 1 will completely determine the arrangement (or permutation) of all 25 (a_i, A_j)'s in Fig. 1. Now the arrangement of the five (a_i, A_j)'s in the top row of Fig. 1 will be determined by the two permutations. $\begin{bmatrix} a_0, & a_1, & a_2, & a_3, & a_4 \\ a_{i0}, & a_{i1}, & a_{i2}, & a_{i3}, & a_{i4} \end{bmatrix}$ and $\begin{bmatrix} A_0, & A_1, & A_2, & A_3, & A_4 \\ A_{j0}, & A_{j1}, & A_{j2}, & A_{j3}, & A_{j4} \end{bmatrix}$.

A basis for the 14,400 member group of similarity mappings that we are dealing with would be $f_{01}, f_{12}, f_{23}, f_{34}$ and $g_{01}, g_{12}, g_{23}, g_{34}$ where $f_{i,i+1}$ is the permutation of Figs. 1, 4 that is defined by interchanging a_i, a_{i+1} and $g_{i,i+1}$ is the permutation of Figs. 1, 4 that is defined by interchanging A_i, A_{i+1}.

We now list these basic permutations.

9	24	3	4	5
6	7	11	1	10
8	12	13	14	18
16	25	15	19	20
21	22	23	2	17

$f_{01}, a_0 - a_1$

1	10	25	4	5
6	7	8	12	2
19	9	13	14	15
16	17	21	11	20
18	22	23	24	3

$f_{12}, a_1 - a_2$

1	2	6	21	5
3	7	8	9	13
11	20	10	14	15
16	17	18	22	12
4	19	23	24	25

$f_{23}, a_2 - a_3$

1	2	3	7	22
14	4	8	9	10
11	12	16	6	15
13	17	18	19	23
21	5	20	24	25

$f_{34}, a_3 - a_4$

24	9	3	4	5
6	7	8	2	12
11	10	20	14	15
23	17	18	19	13
21	22	16	1	25

$g_{01}, A_0 - A_1$

1	25	10	4	5
13	7	8	9	3
11	12	6	16	15
14	24	18	19	20
21	22	23	17	2

$g_{12}, A_1 - A_2$

1	2	21	6	5
4	14	8	9	10
11	12	13	7	17
16	15	25	19	20
3	22	23	24	18

$g_{23}, A_2 - A_3$

1	2	3	22	7
6	5	15	9	10
18	12	13	14	8
16	17	11	21	20
19	4	23	24	25

$g_{34}, A_3 - A_4$

1	2	3	4	5
21	22	23	24	25
16	17	18	19	20
11	12	13	14	15
6	7	8	9	10

h

We discuss the permutation h in a moment. It is obvious that $f_{i,i+1}^2 = f_{i,i+1} \circ f_{i,i+1} = I$ and $g_{i,i+1}^2 = g_{i,i+1} \circ g_{i,i+1} = I, i = 0, 1, 2, 3$, where I is the identity permutation.

Also, $f_{i,i+1} \circ g_{j,j+1} = g_{j,j+1} \circ f_{i,i+1}, i, j \in \{0, 1, 2, 3\}$.

This is obvious from the definitions of $f_{i,i+1}$ and $g_{j,j+1}$ and this can also easily be verified formally by direct calculation. These basic permutations $f_{01}, f_{12}, f_{23}, f_{34}$,$g_{01}, g_{12}, g_{23}, g_{34}$ will generate all of the 14,400 permutations that we defined in Section 5.

If we start with the extremely magic square of Fig. 2 and operate on it in any way by using the above basic permutations $f_{i,i+1}$ and $g_{j,j+1}$ then the image will always be an extremely magic 5×5 square and we can generate from Fig. 2 14,400 extremely magic 5×5

squares by doing this. For example,

$$f_{34}\begin{bmatrix} 8 & 5 & 21 & 17 & 14 \\ 22 & 19 & 13 & 10 & 1 \\ 15 & 6 & 2 & 24 & 18 \\ 4 & 23 & 20 & 11 & 7 \\ 16 & 12 & 9 & 3 & 25 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 21 & 19 & 12 \\ 24 & 17 & 13 & 10 & 1 \\ 15 & 6 & 4 & 22 & 18 \\ 2 & 23 & 20 & 11 & 9 \\ 16 & 14 & 7 & 3 & 25 \end{bmatrix}$$

and this 5×5 matrix is an extremely magic 5×5 square since the sum of the 5 numbers in each of the 120 magic subsets stated in Section 4 is always 65. Of course, this includes the 10 rows and columns and 10 diagonals. We now deal with the permutation h. In Section 2, we defined the dual matrix of Fig. 1 as the matrix where each (a_i, A_j) is changed to (a_j, A_i). This dual matrix will lead to $14,400$ extremely magic 5×5 squares in exactly the same way as the original matrix. It can be shown that these $14,400$ dual magic squares can be obtained by flipping over the original $14,400$ magic squares.

The permutation h is the permutation that defines the dual matrix of Fig. 1. That is, $h(\text{Fig. 1}) = \text{dual matrix}$.

It is obvious that $h^2 = h \circ h = I$.

By the symmetry between the a_i's and A_i's, it is fairly easy to see that the extremely magic 5×5 squares that are generated by the dual matrix will have the same 120 magic 5 element subsets as the original matrix. Also, the $14,400$ similarity mappings of Section 5 for the original magic squares will be exactly the same as the $14,400$ similarity mappings for the dual magic squares. Therefore, we can use the same basis $f_{01}, f_{12}, f_{23}, f_{34}, g_{01}, g_{12}, g_{23}, g_{34}$ for the similarity mappings of the dual magic squares. By direct computation, it is very easy to show that $f_{i,i+1} \circ h = h \circ g_{i,i+1}$ and $g_{i,i+1} \circ h = h \circ f_{i,i+1}$ for $i = 0, 1, 2, 3$.

Therefore, the number of permutations generated by $f_{01}, f_{12}, f_{23}, f_{34}, g_{01}, g_{12}, g_{23}, g_{34}, h$ will be $2 \cdot 14,400 = 28,800$ since we also know that we cannot flip Fig. 1 over if we do not use h. The group structure of these $28,800$ permutations will be a dihedral type group that is generated from $f_{01}, f_{12}, f_{23}, f_{34}, g_{01}, g_{12}, g_{23}, g_{34}, h$ subject to the restrictions of following laws.

1. $f_{i,i+1} \circ f_{i,i+1} = g_{j,j+1} \circ g_{j,j+1} = h \circ h = I.$
2. $f_{i,i+1} \circ g_{j,j+1} = g_{j,j+1} \circ f_{i,i+1}.$
3. $f_{i,i+1} \circ h = h \circ g_{i,i+1}$
4. $g_{i,i+1} \circ h = h \circ f_{i,i+1}.$

Since $g(f_{01}, f_{12}, f_{23}, f_{34}) \cong S_5$ and $g(g_{01}, g_{12}, g_{23}, g_{34}) \cong S_5$ where S_5 is the symmetric group on $\{0, 1, 2, 3, 4\}$, we see from law 2 that $g(f_{01}, f_{12}, f_{23}, f_{34}, g_{01}, g_{12}, g_{23}, g_{34}) \cong S_5 \times S_5$. This fact was also observed in Section 5. When we add h to $f_{01}, f_{12}, f_{23}, f_{34}, g_{01}, g_{12}, g_{23}, g_{34}$ we see that we generate a dihedral type group that has $28,800$ permutations.

7 A Puzzle

Suppose we start with the 5×5 extremely magic square of Fig. 2. However, suppose that we are not aware that Fig. 2 is extremely magic, but we only know that Fig. 2 is extra magic (or panmagic). That is, we are only aware of the fact that the sums of the 5 numbers in each of the 5 rows, 5 columns and 10 generalized diagonals equal 65. Also, suppose that we are given the 8 basic permutations $f_{01}, f_{12}, f_{23}, f_{34}, g_{01}, g_{12}, g_{23}, g_{34}$ of Section 6. If we operate on the Fig. 2 magic square in any arbitrary way by using these permutations $f_{01}, f_{12}, \cdots, g_{34}$, we will always have an extra magic 5×5 square, and we could generate thousands of these extra magic 5×5 squares. However, we would not have the slightest idea why this is true since we are not aware of the big picture concerning extreme magic squares. Since any panmagic 5×5 square is also extremely magic, this puzzle would also work with any panmagic 5×5 square.

8 A Generalization of the Fig. 1 Matrix

In Fig. 1, suppose we now place the 25 ordered pairs $\{(a_i, A_j) : i, j \in \{0, 1, 2, 3, 4\}\}$ in any arbitrary way in the 5×5 matrix. As always, we let a_0, a_1, a_2, a_3, a_4 and A_0, A_1, A_2, A_3, A_4 be any arbitrary permutation of $0, 1, 2, 3, 4$. As always, we use the code $(a_i, A_j)^{\#} = a_i + 5\,(A_j - 1)$ to place the numbers $1, 2, 3, 4, \cdots, 25$ in the 5×5 matrix. Analogous to this paper, we can deal with all permutations of a_0, a_1, a_2, a_3, a_4 and A_0, A_1, A_2, A_3, A_4 which we call $\begin{bmatrix} a_0 \; a_1 \; a_2 \; a_3 \; a_4 \\ a_{i0} a_{i1} a_{i2} a_{i3} a_{i4} \end{bmatrix}$ and $\begin{bmatrix} A_0 \; A_1 \; A_2 \; A_3 \; A_4 \\ A_{j0} A_{j1} A_{j2} A_{j3} A_{j4} \end{bmatrix}$.

As in this paper, we can find 120 magic 5 element subsets whose sum is always 65 and all $14,400$ 5×5 matrixes will have these same 120 magic 5 element subsets. We can also define the dual matrix and we leave it to the reader to study whether the entire theory in this paper holds in general for the dual matrix.

9 A Project for the Reader

We invite the reader to use the permutations $\begin{bmatrix} a_0 a_1 a_2 a_3 a_4 \\ a_0 a_4 a_3 a_2 a_1 \end{bmatrix}$ and $\begin{bmatrix} A_0 A_1 A_2 A_3 A_4 \\ A_0 A_3 A_1 A_4 A_2 \end{bmatrix}$ and observe that the permutation f that we define maps rows and columns of the Fig. 1 matrix onto generalized diagonals and maps generalized diagonals onto rows and columns. We then invite the reader to find other such permutations f of the Fig. 1 matrix that map the 20 rows, columns and generalized diagonals onto the 20 rows, columns and generalized diagonals in various ways.

10 An Application of the Marriage Theorem

The extremely magic 5×5 squares that we have studied are actually much more magic than what we have explained in this paper. We need the marriage theorems to explain the total magicness of these 5×5 squares. We state a variation in Section 11.

Suppose that we choose any one of the 16 configurations of Section 4 (call it a) and place a on the 5×5 extremely magic square in any arbitrary way. Of course, the sum of the 5 numbers in the configuration a is 65. No matter how we choose a and place a on the 5×5 square we can then choose one of the 16 configurations (call it b) and place b on the 5×5 extremely magic square so that a and b are disjoint. Of course, the sum of the 5 numbers in configuration b is also 65. No matter how a and b are chosen and placed on the 5×5 magic square, we can then choose one of the 16 configurations (call it c) and place c on the 5×5 magic square so that c is pairwise disjoint from each of a and b. Of course, the sum of the 5 numbers in configuration c is also 65. No matter how a, b and c are chosen and placed on the 5×5 magic square, we can then choose one of the 16 configurations (call it d) and place d on the 5×5 magic square so that d is pairwise disjoint from each of a, b, c. Of course, the sum of the 5 numbers in d is 65. No matter how a, b, c and d are chosen and placed on the 5×5 magic square, the 5 remaining squares of the 5×5 matrix will be one of the 16 configuration (call it e). Of course, the sum of the 5 numbers in configuration e will be 65. Thus, we have partitioned the 25 squares of the 5×5 magic square into 5 sets a, b, c, d, e and the sum of the 5 numbers in each of a, b, c, d, e is 65. This fact is much more powerful than what we have studied. This can be proved by using the version of Hall's Marriage Theorem stated in section 11, and we consider this to be beyond the scope of this paper.

If a person did not know about the Fig. 1 drawing, the proof of this would be almost hopeless even if one knows the variation of Hall's Marriage Theorem we mention below.

11 A Useful Marriage Theorem

Suppose dots are placed in the squares of an $n \times n$ checkerboard arbitrarily but so that each row and each column has exactly k dots. We allow more than one dot to be placed in a square. Then it is possible to find a subset of n dots so that each row and each column contains exactly one dot.

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References

[1] http://en.wikipedia.org/wiki/Hall

[2] http://www.cut-the-knot.org/arithmetic/elegant.shtml

[3] http://en.wikipedia.org/wiki/Pandiagonal_magic_square