PROBLEM DEPARTMENT

ASHLEY AHLIN* AND HAROLD REITER†

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@email.uncc.edu`. *Electronic submissions using LATEX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by July 1, 2005. Solutions identified as by students are given preference.*

Problems for Solution.

1079. *Proposed by Arthur Holshouser, Charlotte NC.*

Let $(S, *)$ be a set with a binary operation $*$ satisfying for all a, b, c, d in S,

1. $a * b = b * a$,
2. $a * a = a$,
3. $(a * b) * a = b$, and
4. $(a * b) * (c * d) = (a * c) * (b * d)$.

Next let t be an arbitrary member of S and define

$$a + b = t * (a * b).$$

Show that

1. $(S, +)$ is an abelian group with identity t and
2. for every $a \in S$, $(a + a) + a = a + (a + a) = t$.

1080. *Proposed by Brian Smith.*

A deck of nine cards can be numbered so that the sum of the numbers on a randomly chosen pair of cards totals to an integer from 2 to 12 with the same frequency as rolling two standard dice. What are the numbers on the nine cards? Is the solution unique?

1081. *Proposed by Tom Moore, Bridgewater State College, Bridgewater, MA.*

Let t_n be the n'th triangular number, defined by $t_1 = 1$, and $t_n = t_{n-1} + n$ for all $n \geq 2$. Prove that $\gcd(t_{n-1}, t_n) \cdot \gcd(t_n, t_{n+1}) = t_n$ for all $n \geq 2$.

1082. *Proposed by James Rudzinski, Emory University, Atlanta Ga.*

An n-staircase is a grid of $1+2+\cdots+n = \binom{n+1}{2}$ squares arranged so that column 1 has 1 square, column 2 has 2 squares, ..., and column n has n squares. How many *rectangular* regions are bounded by the gridlines of an n staircase? The figure shows

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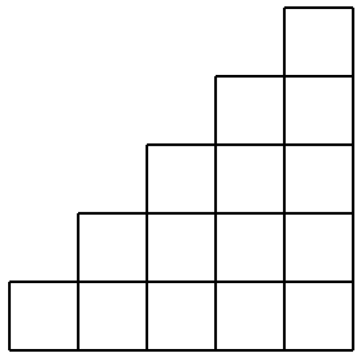
a 5-staircase.

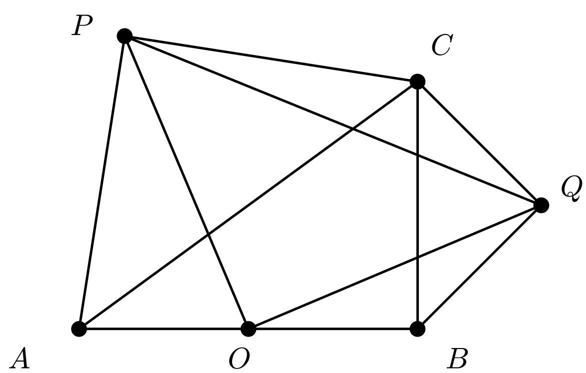


1083. *Proposed by Brian Smith.*

Let N denote a positive integer. Find the number of solutions of $a+b+c=N$ with $a<b<c$, where a, b, and c are positive integers.

1084. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington PA.*

Given triangle ABC, isosceles right triangles ACP and CBQ are constructed, external to $\triangle ABC$, with right angles at P and Q. Prove that if O is the midpoint of AB, then $\angle POQ$ is also a right angle and $\triangle POQ$ is isosceles.



1085. *Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, MI.*

Let $(I_m)_{m\in N}$ be the sequence defined by:

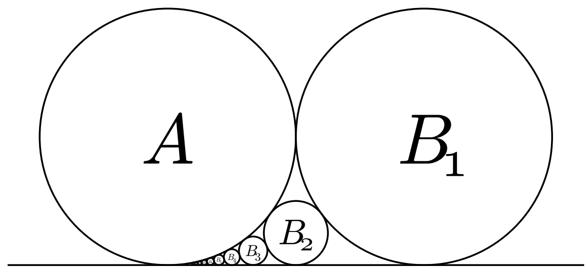
$$I_m = \int_0^1 \frac{x^{2m} e^{\arctan x}}{\sqrt{1+x^2}}\,dx;$$

a. Find a recurrence relation for I_m.
b. Evaluate $\lim_{m\to\infty} mI_m$.

1086. *Proposed by Nathan Bronson, Chapel Hill, NC.*

A and B_1 are tangent circles of radius 1, both tangent to a line as shown. B_{i+1} is a circle tangent to A, B_i, and the line for all $i \geq 1$. The interior of no circles overlap.

What is the sum of the areas of all B_i, $i \geq 1$?



1087. *Proposed by Yuri Godin, University of North Carolina Charlotte.*
Let

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Does there exist a square matrix B such that $B^2 = A$?

1088. *Proposed by Ben Klein, Davidson, NC.*
Let L be the line segment from $(0, 1/2)$ to $(1, 1)$. Fix a positive integer n and let

- $x_k = k/n$ for $0 \leq k \leq n$,
- P_k be the point on L with x-coordinate x_k,
- R_k be the region bounded above by L and below by the unique parabola that passes through the points P_{k-1}, P_k, and $M_k = ((x_k + x_{k-1})/2, 0)$, and
- A_n be the sum of the areas of the R_k's.

Find $\lim_{n\to\infty} A_n$.

1089. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abingtion PA.*

Let ABC be a triangle with an interior point P, and let segments AD, BE and CF be the cevians through P. Let A', B' and C' be the midpoints of the sides BC, CA and AB, respectively. From A', B', C' three cevians $A'D', B'E', C'F'$ are drawn in triangle $A'B'C'$ such that $A'D'||AD, B'E'||BE$ and $C'F'||CF$. Show that

i. $A'D', B'E', C'F'$ meet at a point (call it P')
ii. PP' is divided internally by the centroid G of triangle ABC in the ratio $2:1$.
iii. In the special case when AD, BE and CF are the altitudes of triangle ABC, the line PP' becomes Euler's line.

1090. *Proposed by Robert Gebhardt, Hopatcong, NJ.*
Find the exact value of

$$\sum_{n=1}^{\infty} \frac{n^2(n+2)^2}{(n+1)^6} = \frac{9}{64} + \frac{64}{729} + \frac{225}{4096} + \cdots.$$

1091. *Proposed by Stanislav Molchanov, University of North Carolina Charlotte, Charlotte, NC.*

A regular tetrahedron can be built using six pencil-sized cylinders. When this is done, each of the cylinders touches exactly four of the other cylinders. Is it possible to position six pencil-shaped cylinders so that each one touches the other five?

1092. *Proposed by the editors.* A relation R on a set A is a set of ordered pairs of members of A. That is R is a relation on A if $R \subset A \times A$. A relation R is called *transitive* if $\forall x, y, z \in A, xRy \wedge yRz \Rightarrow xRz$. Let $A = \{1, 2, 3\}$. There are $2^9 = 512$ relations A. How many of these relations are transitive?