PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2012. Solutions identified as by students are given preference.*

Problems for Solution.

1243. *Proposed by M. B. Kulkarni, B. Y. K. College, Nasik India and M. N. Deshpande, Nagpur, India*

We toss an unbiased coin n times. The results of the random experiment are written in a linear array. Suppose R_n and X_n respectively denote the number of runs of like symbols and number of double HH's (overlapping allowed). Show that $Cov(R_n, X_n) = \frac{-n+1}{8}$.

1244. *Proposed by Cecil Rousseau, University of Memphis, Memphis, TN.*

The function $\cos(\log x)$ has an interesting property: its nth derivative is given simply by

$$\frac{d^n}{dx^n}\cos(\log x) = \frac{a_n \cos(\log x) - b_n \sin(\log x)}{x^n},$$

where a_n and b_n are integers.

1. Prove the last statement.
2. Find recurrences for (a_n) and (b_n), and use them to construct a table (n, a_n, b_n) for $n = 1, 2, \ldots, 10$.
3. The unsigned Stirling number of the first kind $\begin{bmatrix} n \\ k \end{bmatrix}$ is the number of permutations of $[n]$ that have k cycles. Use the change of basis identity

$$x^{\underline{n}} = \sum_k \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} x^k,$$

to prove

$$a_n = (-1)^n \sum_k \begin{bmatrix} n \\ 2k \end{bmatrix} (-1)^k \qquad \text{and} \qquad b_n = (-1)^{n+1} \sum_k \begin{bmatrix} n \\ 2k+1 \end{bmatrix} (-1)^k.$$

Provide a combinatorial interpretation of a_n and b_n.

Note. The author gratefully acknowledges the assistance of Floridia Jackson in the early phase of the exploration that led to this problem proposal.

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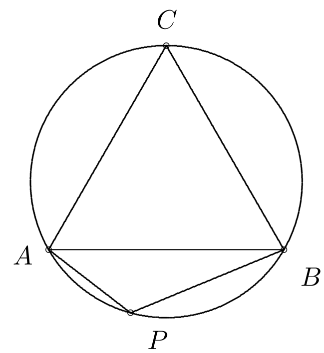
1245. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA)*

Let ABC be an equilateral triangle with edge length c inscribed in a circle. Let P be a point on minor arc $\widehat{AB}$. Let $PA = a$ and $PB = b$. Is it possible for a, b, and c to all be distinct positive integers?



1246. *Proposed by Bessem Samet, Tunis College of Sciences, Tunisia.*

Characterize the set of functions $f : (0, \infty) \to (0, \infty)$ satisfying the following properties:

(a) f is a bounded function,
(b) f is of class C^2 on $(0, +\infty)$, and
(c) $xg''(x) + (1 + xg'(x))g'(x) \geq 0, \ \forall\, x > 0$, where $g(x) = \ln(f(x))$.

1247. *Proposed by Arthur Holshouser, Charlotte, NC and Patrick Vennebush, NCTM, Reston, VA.*

Let P be a $2n$-sided regular polygon. Suppose $k \geq 3$ points are randomly and uniformly selected from the boundary of P. Find the probability that the convex hull of the k points includes the center of P.

1248. *Proposed by José Luis Díaz-Barrero, Universidad Politécnica de Cataluña, Barcelona, Spain.*

Let T_n be the nth triangular number defined by $T_n = \binom{n+1}{2}$ for all $n \geq 1$. Show that for all integers $k \geq 2$, the sequence $\{T_n^k\}_{n \geq 1}$ does not contain any infinite subsequence with all terms in arithmetic progression.

1249. *Proposed by Peter Linnell, Vinginia Polytechnic University, Blacksburg, VA.* This was problem 5 on the annual VPI Regional College Math Contest, 1994.

Let $f\colon \mathbb{Z} \times \mathbb{Z} \to \mathbb{R}$ be a function which satisfies $f(0, 0) = 1$ and

$$f(m, n) + f(m + 1, n) + f(m, n + 1) + f(m + 1, n + 1) = 0$$

for all $m, n \in \mathbb{Z}$ (where $\mathbb{Z}$ and $\mathbb{R}$ denote the set of all integers and all real numbers, respectively). Prove that $|f(m, n)| \geq 1/3$, for infinitely many pairs of integers (m, n).

1250. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA.*

If on the sides of an arbitrary triangle ABC three similar triangles AKB, BLC, and CNA are drawn outward (or inward), then the triangles KLN and ABC have the same centroid G.

1251. *Proposed by Arthur Holshouser, Charlotte, NC.*

This problem appeared on the 2011 Lower Michigan Mathematics Competition.

The operator $(R, \odot)$ on the real numbers R is defined by

$$x \odot y = \frac{xy}{xy + (1-x)(1-y)}$$

when $xy + (1-x)(1-y) \neq 0$. $x \odot y$ is not defined when $xy + (1-x)(1-y) = 0$.

1. Show that $(R, \odot)$ has an identity $I \in R$ such that $x \odot I = I \odot x = x$ for all $x \in R$.
2. Show that each $x \in R \setminus \{0,1\}$ has an inverse x^{-1} such that $x \odot x^{-1} = x^{-1} \odot x = I$.
3. Show that $(R, \odot)$ is both commutative and associative when all operations involved are defined.
4. Show that for all $x \in R$, $x \odot x \odot \cdots \odot x$ (n-times) is always defined and $x \odot x \odot \cdots \odot x$ (n-times) $= \frac{x^n}{x^n + (1-x)^n}$.
5. Discuss the possibility of "patching up" $(R, \odot)$ so that $(R, \odot)$ is a true group.

1252. *Proposed by Neculai Stanciu, Emil Palade Secondary School, Buzau, Romania.*

Let p be a prime not 2 or 5, let a be a digit and let m and n be positive integers. Show that there exist infinitely many numbers of type $A = a \cdot p^n$ whose last m digits are $0 \ldots 0a$.