Win Sequences for Round-robin Tournaments

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Abstract Each of n teams numbered $1, 2, \cdots, n$ play each of the other $n-1$ teams exactly one time for a total of $\binom{n}{2} = n(n-1)/2$ games. Each game produces a win for one team and a loss for the other team. Define $(w_i, l_i), i = 1, 2, \cdots, n$, to be the win-loss records for the teams. That is, team $i, i = 1, 2, \cdots, n$, wins w_i games and loses l_i games where $w_i + l_i = n - 1$. Of course, $\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} l_i = \frac{n(n-1)}{2}$. Suppose $(w_i, l_i), i = 1, 2, \cdots, n$, are arbitrarily specified win-loss records for the teams $1, 2, \cdots, n$ subject only to the conditions $0 \le w_i, 0 \le l_i, w_i + l_i = n - 1$ and $\sum_{i=1}^{n} w_i = \sum_{i=1}^{n} l_i = \frac{n(n-1)}{2}$. In this paper we prove necessary and sufficient conditions that $(w_i, l_i), i = 1, 2, \cdots, n$, must satisfy so that $(w_i, l_i), i = 1, 2, \cdots, n$, is realizable in a tournament.

In section 1 we find necessary conditions on the win sequence w_i. In sections 2 and 3, we prove that these conditions are also sufficient. Our inductive proof gives a short and simple algorithm for constructing a solution. Also this algorithm provides alternate necessary and sufficient conditions since it works if and only if the (w_i, l_i), $i = 1, 2, \ldots, n$, is realizable. In section 4 we prove a fairly obvious equivalent set of necessary and sufficient conditions on $(w_i, l_i), i = 1, 2, \cdots, n$, and in section 5 we briefly discuss extensions of the problem. In [2], Kemnitz and Dolff give an existential proof of our main result. Note that the proof given here is constructive. Other related results can be found in [4] and [1].

Section 1

In this section, we find necessary conditions on $(w_i, l_i), i = 1, 2, \cdots, n$, in order that it be realizable.

Since $0 \leq w_i, 0 \leq l_i, w_i + l_i = n-1, i = 1, 2, \cdots, n$, it is not necessary to specify both w_i and l_i. Therefore, we specify only the win records $w_i, i = 1, 2, \cdots, n$, subject to (1) $0 \leq w_i \leq n-1$, and (2) $\sum\limits_{i=1}^{n} w_i = \frac{n(n-1)}{2}$.

Therefore, if $w_i, i = 1, 2, \cdots, n$, are arbitrarily specified subject only to $0 \leq w_i \leq n-1, \sum\limits_{i=1}^{n} w_i = \frac{n(n-1)}{2}$, we wish to find necessary and sufficient conditions that $w_i, i = 1, 2, \cdots, n$, must satisfy in order to be realizable.

In graphical form, we are given the complete undirected graph K_n on n vertices $1, 2, \cdots, n$. This means that there is one undirected edge between each pair of distinct vertices. The problem requires us to find necessary and sufficient conditions on $w_i, i = 1, 2, \cdots, n$, so that we can place a direction on each edge in such a way that for each vertex $i \in \{1, 2, \cdots, n\}$, the number of directed edges leaving vertex i equals w_i

<u>Necessary Conditions</u> Suppose for $1 \leq k \leq n$ we choose any combination $\{n_1, n_2, \cdots, n_k\}$ of k teams from the collection of n teams. Now these k teams play $\binom{k}{2} = k\,(k-1)\,/2$ games among themselves. Therefore, the total number of wins among themselves for these k teams equals $k\,(k-1)\,/2$. Also, each of the k team plays each of the $n-k$ remaining teams one time for a total of $k\,(n-k)$ games. It follows that condition $(3')$ below is a necessary condition.

$(3')$: For each $1 \leq k \leq n$, any combination of k teams $\{n_1, n_2, \cdots, n_k\}$ satisfy $\sum\limits_{i=1}^{k} w_{n_i} \leq \frac{k(k-1)}{2} + k\,(n-k) = \frac{k}{2}\,(2n-k-1)\,.$

If we agree to write $0 \leq w_1 \leq w_2 \leq \cdots w_n \leq n-1$, then the above necessary condition $(3')$ is equivalent to the following condition (3). (3): $\forall k \in \{1, 2, \cdots, n\}\,, \sum\limits_{n+1-k}^{n} w_i \leq \frac{k}{2}\,(2n-k-1)\,.$

<u>Note</u> The following condition (3^*) is also obviously necessary. However, condition (3^*) is not used in this paper, and indeed in section 5 we easily prove that condition (3^*) is equivalent to condition $(3')$.

(3^*) $\forall k \in \{1, 2, \cdots, n\}\,, \forall \{n_1, n_2, \cdots, n_k\} \subseteq \{1, 2, \cdots, n\}\,, c_2^k = \frac{k(k-1)}{2} \leq \sum\limits_{i=1}^{k} w_{n_i}.$

Section 2

In this section, we rewrite the necessary conditions on $w_i, i = 1, 2, \cdots, n$, developed in section 1 and lay the groundwork for proving that these conditions are also sufficient.

Writing $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq n-1$, this means that we prove the following conditions (1), (2), (3) are both necessary and sufficient for $w_i, i = 1, 2, \cdots, n$, to be realizable.

(1) $0 \leq w_i \leq n-1, i = 1, 2, \cdots, n.$

(2) $\sum\limits_{i=1}^{n} w_i = \frac{n(n-1)}{2}.$

(3) $\forall k \in \{1, 2, \cdots, n\}, \sum\limits_{n+1-k}^{n} w_i \leq \frac{k}{2}(2n-k-1).$

Next, we use induction on n to prove that (1), (2), (3) are sufficient. As always we write $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq n-1$.

It is obvious that conditions (1), (2), and (3) are sufficient for $n = 1, 2$ teams. Therefore, we use induction on n and suppose that the conditions are necessary and sufficient for $1, 2, 3, \cdots, n-1$ teams, where $n-1 \geq 2$. To show that the conditions 1,2, and 3 are sufficient for n teams, we focus our attention on team n. If $w_n \leq n-2$, then $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq n-2$ which implies $0 \leq w_i \leq n-2, i = 1, 2, \cdots, n$. Therefore, each team $i \in \{1, 2, \cdots, n\}$ loses at least one game. Next, suppose $w_n = n-1$. Using $k = 2$, we see that $w_n + w_{n-1} \leq \frac{k}{2}(2n-k-1) = 2n-3$.

Therefore, $w_{n-1} \leq (2n-3)-(n-1) = n-2$. Therefore, $0 \leq w_1 \leq w_2 \leq \cdots \leq w_{n-1} \leq n-2$ which means that $\forall i \in \{1, 2, \cdots, n-1\}, w_i \leq n-2$. This means that each of the teams $1, 2, \cdots, n-1$ loses at least one game.

We now show that each team $i \in \{2, 3, \cdots, n\}$ wins at least one game. If $w_2 \geq 1$, then the conclusion is obvious since $w_2 \leq w_3 \leq \cdots \leq w_n$. Therefore, suppose $w_1 = w_2 = 0$. From (2), $\sum\limits_{i=1}^{n} w_i = \sum\limits_{i=3}^{n} w_i = \frac{n}{2}(n-1)$.

Using $k = n-2$ in (3), we see that $\sum\limits_{i=3}^{n} w_i \leq \frac{k}{2}(2n-k-1) = \left(\frac{n-2}{2}\right)(n+1) < \frac{n}{2}(n-1)$ which is a contradiction. Therefore, each team $i \in \{2, 3, \cdots, n\}$ wins at least one game.

Let us now agree that team n wins a game against w_n of the weakest teams. We use this idea repeatedly in our inductive proof which gives a constructive algorithm.

Since $0 \le w_1 \le w_2 \le \cdots \le w_{n-1} \le w_n \le n-1$, this means that team n satisfies the following condition $(*)$: team n beats teams $1, 2, 3, \cdots, w_n$ and loses against teams $w_n + 1, w_n + 2, w_n + 3, \cdots, n-1$.

We will slightly modify scheme $(*)$ in a moment after we first explain a small problem that arises when we use $(*)$.

Of course, in graphical form, when team n wins against team $i, i = 1, 2, \cdots, n-1$, we draw a directed edge from n to i and when team n loses against team $i, i = 1, 2, \cdots, n-1$, we draw a directed edge from i to n. Let us now delete team n and its $n-1$ wins and losses from consideration. We now focus our attention on teams $1, 2, \cdots, n-1$. We define $\overline{w}_i, i = 1, 2, \cdots, n-1$, to be the wins of each team i from among the remaining teams $\{1, 2, \cdots, n-1\}$. Of course, $\forall i \in \{1, 2, \cdots, n-1\}$, if team n beats team i then $\overline{w}_i = w_i$. On the other hand, if team i beats team n then $\overline{w}_i = w_i - 1$. We now show that $\overline{w}_i,\ i = 1, \ldots, n-1$ satisfy (1), (2), and (3).

Since $1 \le w_2 \le w_3 \le \cdots w_{n-1} \le n-2$, since $0 \le w_1 \le w_2$ and since team n beats team 1 making $\overline{w}_1 = w_1$, it is obvious that (1) is true.

(1) $0 \le \overline{w}_i \le n-2, i = 1, 2, \cdots, n-1.$

We now show that (2) is also true.

(2) $\sum\limits_{i=1}^{n-1} \overline{w}_i = c_2^{n-1} = \frac{(n-1)(n-2)}{2}.$

To see this, observe that team n is involved in exactly $n-1$ wins and losses. Also, team n along with these $n-1$ wins and losses have been deleted from considerations. This gives $\sum\limits_{i=1}^{n-1} \overline{w}_i = \left(\sum\limits_{i=1}^{n} w_i\right) - (n-1) = c_2^n - (n-1) = c_2^{n-1}$.

We must now prove that condition (3) is satisfied for $\overline{w}_1, \overline{w}_2, \cdots \overline{w}_{n-1}$. Since we need to have $0 \le \overline{w}_1 \le \overline{w}_2 \le \cdots \le \overline{w}_{n-1} \le n-2$, we face a slight problem with the scheme $(*)$. In order to maintain $0 \le \overline{w}_1 \le \overline{w}_2 \le \cdots \le \overline{w}_{n-1} \le n-2$, we now slightly modify $(*)$ as follows. This will still mean that team n beats w_n of the weakest teams.

Suppose team n beats w_n teams where $t \leq w_n \leq k$ and $w_{t-1} < w_t = w_{t+1} = \cdots = w_k < w_{k+1}$.

In order to keep $\overline{w}_1 \leq \overline{w}_2 \leq \cdots \leq \overline{w}_{n-1}$, we agree to use scheme $(**)$ in the place of $(*)$. $(**)$: team n beats all of the teams $1, 2, \cdots, t-1$ which is the same as in $(*)$.

However, team n wins its remaining $w_n - (t-1) = w_n - t + 1$ games by beating $w_n - t + 1$ of the teams $t, t+1, t+2, \cdots, k-1, k$ in the reverse order $k, k-1, k-2, \cdots, t+1, t$. Of course, this means that team n beats the teams $k+t-w_n, k+t-w_n+1, \cdots, k-1, k$ and loses to the teams $t, t+1, \cdots, k+t-w_n-1$. This modified scheme $(**)$ which replaces scheme $(*)$ will guarantee that $0 \leq \overline{w}_1 \leq \overline{w}_2 \leq \cdots \leq \overline{w}_{n-1} \leq n-2$. We now state Definition 1 and prove Lemma 1.

Definition 1 Suppose team n beats θ of the teams $n_1, n_2, \cdots, n_t$ where $0 \leq \theta \leq w_n, 0 \leq \theta \leq t$ and $\{n_1, n_2, \cdots, n_t\} \subseteq \{1, 2, \cdots, n-1\}$. Then $x = t - \theta$ is called the *skip* of team n with respect to $\{n_1, n_2, \cdots, n_t\}$. In other words, we say that team n *skips* over x of the teams $n_1, n_2, \cdots, n_t$.

Lemma 1 As always, team n wins w_n games against the teams $1, 2, \cdots, n-1$. As always, let us delete team n and as always let $\overline{w}_1, \overline{w}_2, \cdots, \overline{w}_{n-1}$ be the number of games that teams $1, 2, \cdots, n-1$ win among themselves.

Suppose $\{n_1, n_2, \cdots, n_t\} \subseteq \{1, 2, \cdots, n-1\}$ and suppose team n skips over x of the teams $\{1, 2, \cdots, n-1\} \setminus \{n_1, n_2, \cdots, n_t\}$ where, of course, $0 \leq x \leq n-1-t$.

Then $\left(\sum_{i=1}^{t} w_{n_i}\right) + x + w_n = \left(\sum_{i=1}^{t} \overline{w}_{n_i}\right) + n - 1.$

Proof. Let $w_n = w^*_{n_0} + w^*_{n_1}$ where $w^*_{n_0}$ is the number of the teams in the set $\{1, 2, \cdots, n-1\} \setminus \{n_1, n_2, \cdots, n_t\}$ that team n beats and $w^*_{n_1}$ is the number of the teams $\{n_1, n_2, \cdots, n_t\}$ that team n beats. Of course, $w^*_{n_0} + x = n-1-t$ since $n-1-t$ is the number of elements in $\{1, 2, \cdots, n-1\} \setminus \{n_1, n_2, \cdots, n_t\}$. Also, since $\{n_1, n_2, \cdots, n_t\}$ has t elements, we see that $\left(\sum_{i=1}^{t} w_{n_i}\right) + w^*_{n_1} = \left(\sum_{i=1}^{t} \overline{w}_{n_i}\right) + t$, since $\sum_{i=1}^{t} w_{n_i} - \sum_{i=1}^{t} \overline{w}_{n_i} = t - w_{n_1}$.

Therefore,

$$\begin{aligned}\left(\sum_{i=1}^{t} w_{n_i}\right)+x+w_n &= \\ \left(\sum_{i=1}^{t} w_{n_i}\right)+x+w_{n_0}^*+w_{n_1}^* &= \\ \left[\left(\sum_{i=1}^{t} w_{n_i}\right)+w_{n_1}^*\right]+\left(w_{n_0}^*+x\right) &= \\ \left[\left(\sum_{i=1}^{t} \overline{w}_{n_i}\right)+t\right]+(n-1-t)=\left(\sum_{i=1}^{t} \overline{w}_{n_i}\right)+n-1.\end{aligned}$$

$\square$

Section 3

In this section, we finish the proof that the conditions on $w_i, i=1,2,\cdots,n$ are necessary and sufficient.

As before, we will let team n beat w_n of the teams. Using the modified scheme $(**)$, we are guaranteed that $0 \leq \overline{w}_1 \leq \overline{w}_2 \leq \cdots \leq \overline{w}_{n-1} \leq n-2$ will be true after team n is deleted. By induction, the proof is complete if we can prove (3).

(3) $\forall k \in (1,2,\cdots,n-1)\,, \sum_{n-k}^{n-1} \overline{w}_i \leq c_2^k + k\,(n-1-k) = \frac{k}{2}\,(2n-k-3)$. Considering k to be fixed; we now consider 3 cases.

<u>Case 1</u> $w_n \leq n-1-k$.

<u>Case 2</u> $w_n \geq n-k$ and $skip\,(\{1,2,\cdots,n-1-k\})=0$.

<u>Case 3</u> $w_n \geq n-k$ and $skip\,(\{1,2,\cdots,n-1-k\}) \geq 1$.

We consider Case 2 separately from Case 3 so that the reader will gain more insight.

<u>Case 1</u> Since $0 \leq w_1 \leq w_2 \leq \cdots \leq w_n \leq n-1-k$, we have $\sum_{n-k}^{n-1} \overline{w}_i \leq \sum_{n-k}^{n-1} w_i \leq k\,(n-1-k) \leq c_2^k + k\,(n-1-k)$, which is what we need to prove.

<u>Case 2</u> By the inductive hypothesis, we know that (a) $\sum_{n-k}^{n} w_i = \left(\sum_{n-k}^{n-1} w_i\right) + w_n \leq$

$c_2^{k+1} + (k+1)(n-1-k) = \left(\frac{k+1}{2}\right)(2n-k-2)$.

As always, we need to show that $\sum_{n-k}^{n-1} \overline{w}i \leq c_2^k + k(n-1-k) = \frac{k}{2}(2n-k-3)$.

Now in Case 2, $x = skip(\{1, 2, \cdots, n-1-k\}) = 0$. Using Lemma 1 with $x = 0$, we see that $\left(\sum_{n-k}^{n-1} w_i\right) + 0 + w_n = \left(\sum_{n-k}^{n-1} \overline{w}_i\right) + (n-1)$. Therefore, from (a) $\left(\sum_{n-k}^{n-1} \overline{w}_i\right) + n - 1 \leq \left(\frac{k+1}{2}\right)(2n-k-2)$.

Therefore, $\sum_{n-k}^{n-1} \overline{w}_i \leq \left(\frac{k+1}{2}\right)(2n-k-2) - (n-1) = \frac{k}{2}(2n-k-3)$.

<u>Case 3</u> In Case 3, $x = skip(\{1, 2, \cdots, n-1-k\}) \geq 1$, and also $w_n \geq n-k$.

Since $w_n \geq n-k$, it is fairly obvious from the way that scheme (**) is defined that the following equality-inequality must be true. This is the only way that team n can skip x of the teams $\{1, 2, \cdots, n-1-k\}$. This equality-inequality does not tell and we do not care precisely which x of the teams $\{1, 2, \cdots, n-1-k\}$ team n skips.

$$\begin{aligned} 0 &\leq x_1 \leq \cdots \leq x_{n-k-x-1} \leq w_{n-k-x} = w_{n-k-x+1} = \cdots = w_{n-k-1} = w_{n-k} \\ &= w_{n-k+1} = \cdots = w_{n-k+x-1} \leq w_{n-k+x} \leq \cdots \leq w_{n-1}. \end{aligned}$$

Let us call $w_{n-k-x} = w_{n-k-x+1} = \cdots = w_{n-k+x-1} = w$.

As always, we again need to show that $\sum_{n-k}^{n-1} \overline{w}_i \leq c_2^k + k(n-1-k) = \frac{k}{2}(2n-k-3)$.

We now prove Lemma 2 which will allow us to prove this inequality almost exactly as we did in Case 2 by simply substituting x for 0 in the case 2 proof.

Lemma 2 Suppose $w_{n-k-x} = \cdots = w_{n-k-1} = w_{n-k} = w_{n-k+1} = \cdots = w_{n-k+x-1} = w$.

Then $\sum_{n-k}^{n} w_i = \left(\sum_{n-k}^{n-1} w_i\right) + w_n \leq c_2^{k+1} + (k+1)(n-k-1) - x = \left(\frac{k+1}{2}\right)(2n-k-2) - x$.

Proof. Let us suppose the conclusion is false. That is, let us assume that

(a) $\sum_{n-k}^{n} w_i \geq \left(\frac{k+1}{2}\right)(2n-k-2) - x + 1$.

We show that (a) leads to a contradiction. By the inductive hypothesis, we know that

$$\left(\sum_{n-k}^{n} w_i\right) + \left(\sum_{n-k-x}^{n-k-1} w_i\right)$$
$$= \left(\sum_{n-k}^{n} w_i\right) + xw \leq c_2^{k+1+x} + (k+1+x)(n-k-1-x)$$
$$= \left(\frac{k+1+x}{2}\right)(2n-k-x-2)\,.$$

Using (a) with this inequality we see that (b) is true.

(b) $xw \leq \left(\frac{k+1+x}{2}\right)(2n-k-x-2) - \sum_{n-k}^{n} w_i \leq \left(\frac{k+1+x}{2}\right)(2n-k-x-2) - \left(\frac{k+1}{2}\right)(2n-k-2) + x - 1$.

Observe that

(c) $\sum_{n-k+x}^{n} w_i \leq c_2^{k-x+1} + (k-x+1)(n-k+x-1) = \left(\frac{k-x+1}{2}\right)(2n-k+x-2)\,.$

Using (b) and (c) we see that (d) is true.

(d)

$$\sum_{n-k}^{n} w_i = \sum_{n-k+x}^{n} w_i + \sum_{n-k}^{n-k+x-1} w_i$$
$$= \left(\sum_{n-k+x}^{n} w_i\right) + xw \leq \left(\frac{k-x+1}{2}\right)(2n-k+x-2)$$
$$+ \left(\frac{k+1+x}{2}\right)(2n-k-x-2) - \left(\frac{k+1}{2}\right)(2n-k-2) + x - 1.$$

We now show that (a) and (d) are incompatible which proves that assumption (a) must be false.

Therefore, we must show that the right expression in (d) is less than the right expression in (a) which is the following: $\left(\frac{k-x+1}{2}\right)(2n-k+x-2) + \left(\frac{k+1+x}{2}\right)(2n-k-x-2) - \left(\frac{k+1}{2}\right)(2n-k-2) + x - 1 < \left(\frac{k+1}{2}\right)(2n-k-2) - x + 1.$

By collecting like terms and then multiplying by 2, this is true if and only if $(k-x+1)(2n-k+x-2)+(k+1+x)(2n-k-x-2)+4x<2(k+1)(2n-k-2)+4$.

Multiplying out, this is true if and only if

$$\begin{aligned}
& \left[2nk-2nx+2n-k^2+xk-k+xk-x^2+x-2k+2x-2\right] \\
& +\left[2nk+2n+2nx-k^2-k-xk-xk-x-x^2-2k-2-2x\right] \\
+4x \quad < \quad & 4nk+4n-2k^2-2k-4k-4+4.
\end{aligned}$$

After simplifying the above inequality is equivalent to $-2x^2+4x<4$ which is equivalent to $-(x-1)^2<1$ which is always true. $\square$

Lemma 2 now allows us to finish Case 3 almost exactly as we finished Case 2.

As always, we must prove that $\sum_{n-k}^{n-1}\overline{w}_i \leq c_2^k + k(n-1-k) = \frac{k}{2}(2n-k-3)$. Since $x = skip(\{1,2,\cdots,n-k-1\})$, we know from Lemma 1 that $\left(\sum_{n-k}^{n-1} w_i\right)+x+w_n = \left(\sum_{n-k}^{n-1}\overline{w}_i\right)+(n-1)$. From Lemma 2 we know that $\left(\sum_{n-k}^{n-1} w_i\right)+w_n \leq \left(\frac{k+1}{2}\right)(2n-k-2)-x$. Therefore, $\left(\sum_{n-k}^{n-1} w_i\right)+w_n+x \leq \left(\frac{k+1}{2}\right)(2n-k-2)$ which implies $\left(\sum_{n-k}^{n-1}\overline{w}_i\right)+(n-1) \leq \left(\frac{k+1}{2}\right)(2n-k-2)$. Therefore, $\left(\sum_{n-k}^{n-1}\overline{w}_i\right) \leq \left(\frac{k+1}{2}\right)(2n-k-2)-(n-1) = \frac{k}{2}(2n-k-3)$.

This completes Case 3 which finishes the proof. $\square$

Section 4

In this section, we find equivalent necessary and sufficient conditions.

Lemma 3 Conditions (1), (2), ($3'$) of Section 1 are equivalent to (1), (2), (3^*).

(3^*) $\forall k \in \{1,2,\cdots,n\}$, $\forall$ combination of k teams $I_k=\{n_1,n_2,\cdots,n_k\}\subseteq\{1,2,\cdots,n\}$, $\sum_{i\in I_k} w_i = \sum_{i=1}^{k} w_{n_i} \geq c_2^k = \frac{k(k-1)}{2}$.

Proof. We show that (1), (2), ($3'$)$\Rightarrow$(3^*). The proof that (1), (2), (3^*)$\Rightarrow$($3'$) is similar and is left to the reader.

Consider $I_k^c = \{1, 2, \cdots, n\} \setminus \{n_1, n_2, \cdots, n_k\}$.

Also, observe that $c_2^{n-k} + c_2^k + k\,(n-k) = c_2^n$.

Now from (2), $\sum\limits_{i=1}^{n} w_i = \sum\limits_{i \in I_k} w_i + \sum\limits_{i \in I_k^c} w_i = c_2^n$.

From (3′), $\sum\limits_{i \in I_k^c} w_i \leq c_2^{n-k} + (n-k)\,k$.

Therefore, $\sum\limits_{i \in I_k} w_i = c_2^n - \sum\limits_{i \in I_k^c} w_i \geq c_2^n - \left[c_2^{n-k} + (n-k)\,k\right] = c_2^k$. □

Section 5

In this section we point to future directions.

An obvious generalization is to have n teams with each pair of teams playing exactly k games together with each game producing a winner and a loser. We again ask whether a given sequence of win-loss records is realizable. Another generalization is to allow any game to produce one of 3 outcomes namely win, loss, tie. We then ask whether a given sequence of win, loss, tie records is realizable. Another generalization is to have n teams with each game involving 3 teams say. Also, each combination of 3 teams plays exactly k games together. The possible outcome of a game can be defined in different ways. For example, a game may always end with one winner and two losers, or a game may always end with one team winning, one team losing and one team neither winning nor losing. Thus the problem solved in this paper can be extended in many ways.

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