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PROBLEM DEPARTMENT

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This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@email.uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2005. Solutions identified as by students are given preference.*

Problems for Solution.

1093. *Proposed by Peter A. Lindstrom, Batavia, NY.*

Prove that for any integer $n \geq 3$, the last $n+1$ digits of 5^{2^n} are the same as the last $n+1$ digits of $5^{2^{n-1}}$.

1094. *Proposed by Douglas Shafer, University of North Carolina Charlotte, Charlotte, NC.*

Let S and T denote any disjoint pair of countable subsets of the plane. Prove that there exists a family F of parallel lines such that each point of S belongs to exactly one member of F and no member of T belongs to any member of F.

1095. *Proposed by Stanislav Molchanov, University of North Carolina Charlotte, Charlotte, NC.*

Call a quadrilateral *cyclic* if it can be inscribed in a circle that contains all four vertices. A quadrilateral that can be both inscribed and circumscribed on some pair of circles is known as a *bicentric* quadrilateral. If $ABCD$ is a bicentric quadrilateral with sides of length a, b, c and d, in that order, find a function $f(a, b, c, d)$ whose value is the distance between the centers of the two circles. Is it possible to find the distance between the centers of a bicentric quadrilateral given only the radii R and r of the two circles?

1096. *Proposed by Bill Butler, Durango, CO.*

Solve the cryptarithm $VISIT + LAS + VEGAS + SLOTS = LOSE + WAGES$, where each letter represents a single digit and no two letters represent the same digit.

1097. *Proposed by José Luis Díaz-Barrero, Universitat Politècnica de Catalunya, Barcelona, Spain.*

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Evaluate

$$\prod_{k=1}^{\infty}\left(2\cos\frac{\pi}{2^{k+1}}-1\right).$$

1098. *Proposed by Karl David, Milwaukee School of Engineering, Milwaukee, WI.*

Find all real numbers x such that $\sin(x \text{ radians}) = \sin(x \text{ degrees})$. Repeat for $\cos x$ and $\tan x$.

1099. *Proposed by Paul S. Bruckman, Sointula, BC.*

Given that angles A, B, C and D, measured in degrees, are in arithmetic progression with $A < B < C < D < 90$ and that the angles A, B, C are the angles of a triangle, solve the equation

$$\tan A \cdot \tan B \cdot \tan C = \tan D.$$

1100. *Proposed by Stanislav Molchanov, University of North Carolina Charlotte, Charlotte, NC.*

Among all tetrahedra with fixed triangular base and fixed altitude, find the one with least surface area.

1101. *Proposed by Jerry Grossman, Oakland University, Rochester, MI and Gary Thompson, Grove City College, Grove City, PA.*

Ten friends organize a gift exchange. The ten names are put into a hat, and the first person draws one. If she picks her own name, then she returns it to the bag and draws again, repeating until she has a name that is not her own. Then the second person draws, again returning his own name if it is drawn. This continues down the line. What is the probability that when the tenth person draws, only her own name will be left in the bag?

1102. *Proposed by Peter A. Lindstrom, Batavia, NY.*

Consider the function defined by

$$f(k)=\sum_{i=1}^{k}\left(1-\frac{i^2}{k^2}\right)\left(\frac{1}{k}\right),$$

where k is a positive integer. For any positive integer n, show that $f(25\cdot 10^{n-1}) = 0.666\ldots664666\ldots664$, which is a decimal having n 6's followed by a 4 followed by n 6's followed by a 4.

1103. *Proposed by Cecil Rousseau, University of Memphis, Memphis. TN.*

(a) Determine

$$\sum_{n=0}^{\infty}\binom{2n+1}{n}^{-1}.$$

(b) Generalize to

$$\sum_{n=0}^{\infty}\binom{2n+2k+1}{n+k}^{-1}.$$

The sum of the generalized series can be expressed as a finite sum of known numbers.

1104. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA.*

Let ABC be a triangle in which $AB = AC$ and $m\angle A = 100°$. AC is extended to D so that $AD = BC$. Show that $m\angle ADB = 30°$.

1105. *Proposed by Titu Andreescu, University of Texas at Dallas, Arlington, TX.*

Consider the sequences $a_n, n = 0, 1, 2, \ldots$, and $b_n, n = 0, 1, 2, \ldots$ defined by $a_0 = 0, a_1 = 2, a_{n+1} = 4a_n + a_{n-1}, n \geq 1$ and $b_0 = 0, b_1 = 1, b_{n+1} = a_n - b_n + b_{n-1}, n \geq 1$. Prove that $a_n^3 = b_{3n}$ for all n.

1106. *Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland.*

Can anybody reasonably assert that 31000 and 133100 are the same number? Believe it: in a sense, they are. Let their common value be nice and lucky and happy for you! Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the first of several such problems we plan for this column.

1107. *Proposed by Stanislav Molchanov, University of North Carolina Charlotte, Charlotte, NC and Arthur Holshouser, Charlotte, NC.*

Among all hexagons with sides of lengths $561, 561, 884, 884, 1020$, and 1020 what is the area of the one with largest area?

1108. *Proposed by Paul Wrayno, undergraduate, Duke University*

Let A be a regular octahedron with edge length 1. Note that such an octahedron could be formed by starting with a regular tetrahedron of side length 2, and removing the four smaller tetrahedra of side length 1 from each corner as shown. Find the ratio between the radius of the largest sphere that can be inscribed in the octahedron and the radius of the smallest sphere that can be circumscribed about it.