PROBLEM DEPARTMENT

ASHLEY AHLIN* AND HAROLD REITER†

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@email.uncc.edu`. *Electronic submissions using LATEX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by October 1, 2006. Solutions identified as by students are given preference.*

Problems for Solution.

1123. *Proposed by Mike Pinter, Belmont University, Nashville, TN*

This problem is in honor of the 300^{th} birthday of Benjamin Franklin. Consider the base 9 cryptarithm $K\ I\ T\ E\ +\ K\ E\ Y\ =\ S\ H\ O\ C\ K$. Find a solution that minimizes the $S\ H\ O\ C\ K$.

1124. *Proposed by Paul S. Bruckman, Sointula, BC, Canada*

Given positive integers a and b, let $S(a,b) = \sum_{j=0}^{a}\{\lfloor b(1-j^2/a^2)^{1/2}\rfloor + 1\}$. Prove that $S(a,b) = S(b,a)$.

1125. *Proposed by David Wells, Penn State New Kensington, Upper Burrell, PA*

For each positive integer n, let $P(n)$ be the product of the decimal digits of n, let $P_1(n) = P(n)$, and for $k \geq 2$, let $P_k(n) = P(P_{k-1}(n))$. Prove that $P_k(n) = 1$ for some k if and only if n contains no digits other than 1.

1126. *Proposed by Stanley Rabinowitz, MathPro Press, Chelmsford, MA*

Find a rational function $f(x)$ with integer coefficients such that

$$\cos\theta = f(\sin\theta - \cos\theta)$$

is an identity or prove that no identity of this form exists.

1127. *Proposed by Arthur Holshouser, Charlotte, NC*

A bug starts from the origin on the plane and crawls one unit upwards to $(0,1)$ after one minute. During the second minute, it crawls two units to the right ending at $(2,1)$. Then during the third minute, it crawls three units upward, arriving at $(2,4)$. It makes another right turn and crawls four units during the fourth minute. From here it continues to crawl n units during minute n and then making a $90°$, either left or right. The bug continues this until, after 16 minutes, it finds itself back at the

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origin. Its path does not intersect itself. What is the maximum possible area of the 16-gon traced out by its path?

1128. *Proposed by Brian Bradie, Christopher Newport University, Newport News, VA*

Evaluate

$$\int_0^{\pi/2} \frac{1}{1+\tan^n x}\,dx,$$

for $n = 0, 1, 2, \ldots$.

1129. *Proposed by Arthur Holshouser and Stas Molchanov, Charlotte, NC*

Let R denote the real numbers and Q the rational numbers. A function f has a local minimum at the point x_0 if there exists an open neighborhood U of x_0 such that $f(x_0) \le f(x)$ for all $x \in U$.

1. Find a non-constant function $f : R \to R$ such that f has a local minimum at every point.
2. Find a function $g : Q \to Q$ such that for each rational number r, there is neighborhood U of r such that $g(r) < g(x)$ for each $x \in U$.

1130. *Proposed by Marcin Kuczma, University of Warsaw, Warsaw, Poland*

Twins is the keyword for this season. Let $t(1) = 5$, $t(2) = 7$, $t(3) = 13$, $t(4) = 19, \ldots$ be the increasing sequence (finite or infinite?) of all primes such that, for each i, $t(i) - 2$ is also a prime –and let $t(t(t(2)))$ be nice and lucky and happy for you! As usual, the problem is 'what is the year?' Editor's note: this puzzle was sent to friends of the poser in December of a certain year as a gift. This is the third of several such problems we plan for this column.

1131. *Proposed by Ayoub B. Ayoub, Pennsylvania State University, Abington College, Abington, PA*

$ABCD$ is a convex quadrilateral in which ΔBCD is equilateral and $m\angle DAB = 30°$. Show that $(AC)^2 = (AD)^2 + (AB)^2$.

1132. *Proposed by Leo Schneider, John Carroll University, Cleveland, OH*

The two parallel sides of a trapezoid are of length a and b. A segment of length m parallel to these two sides divides the trapezoid into two trapezoids, each of area equal to one half of the original trapezoid. Prove that if a, b, and m are relatively prime positive integers, then neither 2 nor 3 is a prime factor of any of these integers.

1133. *Proposed by Arthur Holshouser, Charlotte, NC; Anita Chatelain and Joe Albree, Auburn University at Montgomery, Montgomery, AL*

In Pillow Problem 14, Lewis Carroll proved in his head that 3 times the sum of 3 squares is also the sum of 4 squares. (Of course, 0^2 is considered to be a square).

1. Prove pillow problem 14.
2. Prove that $\left(a^2+b^2+c^2\right)\left(x^2+y^2+z^2\right)$ is also the sum of 4 squares.
3. Prove that $\prod_{i=1}^{n}\left(x_i^2+y_i^2+z_i^2+v_i^2\right)$ is also the sum of 4 squares by first proving that $\left(a^2+b^2+c^2+d^2\right)\left(x^2+y^2+z^2+v^2\right)$ is the sum of 4 squares.

1134. *Proposed by Paul S. Bruckman, Sointula, BC*

For all $x > 0$, let $\pi(x)$ denote the number of positive integers less than or equal to x that are prime. Prove the following inequality :

$$\pi(2n+4) > 1 + \frac{3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n+1)}{2 \cdot 4 \cdot 6 \cdot \ldots 2n}$$

$n = 1, 2, \ldots$.

1135. *Proposed by Cecil Rousseau, University of Memphis, Memphis, TN*

Let $\phi = (1+\sqrt{5})/2$ denote the golden ratio. Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 |\sin(n\pi\phi)|}$$

converges.

1136. *Proposed by Stas Molchanov, University of North Carolina Charlotte*

Four planar circles are pair-wise externally tangent. Three of the circles are also tangent to a line L. If the radius of the fourth circle is one unit, what is the distance of its center from L?

1137. *Proposed by Peter Lindstrom, Batavia, NY*

Let n be a positive integer and T_i be the i^{th} triangular number.
Find the value of

$$\lim_{k\to\infty} \sum_{i=1}^{k} \frac{\sum_{j=1}^{n} \binom{n}{j} i^{n-j}}{(T_i)^n}.$$