

Mathematics Physical Sciences Doctor of Philosophy

The Large Scale Geometry of Nilpotent-by-Cyclic Groups

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June 2002

Abstract

A nonpolycyclic nilpotent-by-cyclic group Γ can be expressed as the HNN extension of a finitely-generated nilpotent group N . The first main result is that quasi-isometric nilpotent-by-cyclic groups are HNN extensions of quasi-isometric nilpotent groups. The nonsurjective injection defining such an extension induces an injective endomorphism ϕ of the Lie algebra \mathfrak{g} associated to the Lie group in which N is a lattice. A normal form for automorphisms of nilpotent Lie algebras—*permuted absolute Jordan form*—is defined and conjectured to be a quasi-isometry invariant. We show that if ϕ, θ are endomorphisms of lattices in a fixed Carnot group G , and if the induced automorphisms of \mathfrak{g} have the same permuted absolute Jordan form, then $\Gamma_\phi, \Gamma_\theta$ are quasi-isometric. Two quasi-isometry invariants are also found:

- The set of “divergence rates” of vertical flow lines, \mathcal{D}_ϕ
- The “growth spaces” $\mathfrak{g}_n \subset \mathfrak{g}$

These do not establish that permuted absolute Jordan form is a quasi-isometry invariant, although they are major steps toward that conjecture.

Furthermore, the quasi-isometric rigidity of finitely-presented nilpotent-by-cyclic groups is proven: any finitely-presented group quasi-isometric to a non-polycyclic nilpotent-by-cyclic group is (virtually-nilpotent)-by-cyclic.

Acknowledgements Chief thanks are due to my Heavenly Father, who “humbled me, causing me to hunger, and then fed me with manna,” and who through Jesus has generously provided life and breath and the abilities and opportunities to complete this dissertation.

I owe many thanks to Benson Farb for his outstanding mathematical mentoring and faithful encouragement. I also appreciate the mathematical assistance of Chris Connell, John Franks, Lee Mosher, Kevin Whyte, and Dave Witte.

Thanks to Mike Mihalik and the mathematics department at Vanderbilt University for hosting me while I completed my dissertation.

This dissertation came into being in part due to the prayers and encouragement of many, including Chris Ahlin, Harold and Betty Reiter, and the

University of Chicago Graduate Christian Fellowship. I greatly appreciate their investment in this work.

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	Introduction	

The large-scale geometry of a group is captured by the notion of quasi-isometry. A map f between metric spaces X and Y is a *quasi-isometry* if there are constants $K, C, C' \geq 0$ such that:

- for all $x, y \in X$,

$$\frac{1}{K}d_X(x, y) - C \leq d_Y(f(x), f(y)) \leq Kd_X(x, y) + C,$$

and

- the C' neighborhood of $f(X)$ is all of Y .

Given a finitely-generated group G , suppose that S_1 and S_2 are two generating sets for G . Let d_i be the word metric on G induced by S_i . Then (G, d_1) and (G, d_2) are quasi-isometric. Thus, the quasi-isometry type of a group is independent of choice of generating set. In 1980, Gromov established [G81] that the class of finitely-generated nilpotent groups is *quasi-isometrically rigid*; that is, a group is quasi-isometric to a nilpotent group if and only if it is virtually nilpotent. One component of this proof was Wolf's result [Wo] that nilpotent groups have polynomial growth, which is a quasi-isometry invariant and thus a first step towards classifying nilpotent groups up to quasi-isometry. Furthermore, Bass [Bas] found a formula for the precise degree of polynomial growth of a nilpotent group, in terms of its lower central series. These results inaugurated the project of studying the large-scale geometry of solvable groups via quasi-isometries.

One of the first major results in this area was due to Farb and Mosher, who considered the solvable Baumslag-Solitar groups $BS(1, n)$ [FM98]. They established that this class is quasi-isometrically rigid and gave a complete classification up to quasi-isometry. These groups are the simplest examples of a rich class of (typically) non-nilpotent solvable groups called nilpotent-by-cyclic groups. A group Γ is *nilpotent-by-cyclic* if there is an exact sequence

$$1 \rightarrow N \rightarrow \Gamma \rightarrow \mathbb{Z} \rightarrow 1$$

where N is a nilpotent group.

A finitely-generated nilpotent-by-cyclic group Γ can be thought of in a different way. It is also an ascending HNN extension of a nilpotent group N defined by the endomorphism ϕ . If N has presentation $N = \langle R \mid S \rangle$, then Γ is given by the presentation:

$$\Gamma = \langle R, t \mid S, tnt^{-1} = \phi(n), \text{ for } n \in N \rangle.$$

By a result of Bieri and Strebel [BS], if Γ is finitely presented, then it can be represented as the ascending HNN extension of a (possibly different) nilpotent group which is finitely generated. The group Γ is nonpolycyclic if ϕ is not surjective.

The case when N is abelian, i.e., the *abelian-by-cyclic* groups, was considered by Farb and Mosher [FM00] in 1999. Defined like nilpotent-by-cyclic groups,

finitely-generated abelian-by-cyclic groups are the HNN extensions of finitely-generated abelian groups. The action of the nonsurjective injection can be specified by an $n \times n$ integer matrix M . The *absolute Jordan form* of a matrix is obtained from its Jordan form by replacing each diagonal entry with its absolute value. They showed that two finitely-generated, nonpolycyclic abelian-by-cyclic groups are quasi-isometric if and only if the matrices defining them have integral powers with the same absolute Jordan form. Furthermore, they showed that this class of groups is quasi-isometrically rigid. That is, any group which is quasi-isometric to a group in this class has a quotient by a finite subgroup which is (virtually) one itself. The questions of classification and rigidity of nilpotent-by-cyclic groups were later posed by Farb and Mosher ([FM00a], Problem 3 and Question 2, respectively).

1 Statement of Results

The first main result of this dissertation is:

Theorem 1 (Rigidity). *Let $\Gamma = \Gamma_{N,\phi}$ be a finitely-presented nonpolycyclic nilpotent-by-cyclic group. If G is a finitely-generated group quasi-isometric to Γ , then G is the ascending HNN extension of a virtually nilpotent group.*

To begin classifying nilpotent-by-cyclic groups up to quasi-isometry, we first show that two quasi-isometric nilpotent-by-cyclic groups must be based on nilpotent groups which are themselves quasi-isometric. The classification of nilpotent groups up to quasi-isometry is still a major open question. For the purposes of classifying groups up to quasi-isometry, we will restrict our attention to lattices in *Carnot groups*, which are nilpotent Lie groups with a particularly nice nilpotent grading. Lattices in Carnot groups are quasi-isometric if and only if they are lattices in the same Carnot group. Thus, for the remainder of the work on classification, we make the slightly stronger assumption that the base nilpotent groups are lattices in the same Carnot group. We also restrict our attention to nilpotent-by-cyclic groups defined by endomorphisms which act without unipotent part. (See Section 4 for a precise definition.)

Every finitely-generated nilpotent group has a finite-index torsion-free subgroup ([Baum], Theorem 2.1, citing Hirsch), to which it must therefore be quasi-isometric. Also, by a result of Malcev ([M]; see also [R], Theorem 2.18) every finitely-generated torsion-free nilpotent group N is a lattice in a connected, simply-connected nilpotent Lie group G . Malcev also shows that any (injective) endomorphism of the discrete nilpotent group N extends to an (injective) endomorphism of the Lie group G . Given such an endomorphism ϕ , consider the induced linear map ϕ^* which acts on the Lie algebra \mathfrak{g} . We have restricted our attention to nilpotent-by-cyclic groups based on nilpotent groups which are lattices in the same Carnot group G . Thus, we are comparing maps of a fixed Lie algebra \mathfrak{g} .

Such a map can be represented by some matrix $M \in \mathrm{GL}_n(\mathbb{C})$ which is in Jordan form. If all the eigenvalues of M are real, then this matrix represents

the map with respect to some basis $\{e_i\}$. (If M has complex eigenvectors, we carry out a similar procedure to what follows.) Associated to the basis $\{e_i\}$ is a weight vector $w \in \mathbb{N}^n$ which specifies the component of the nilpotent grading in which each vector appears. Given this weight data, we permute the basis vectors to obtain a matrix in *permuted Jordan form*. From this form, we obtain the *permuted absolute Jordan form* of M by replacing each diagonal entry with its absolute value. (See Section 6 for precise definitions.)

This form is a classifier of nilpotent-by-cyclic groups; that is, groups which have the same permuted absolute Jordan form are quasi-isometric.

Theorem 2 (Permuted Absolute Jordan Form). *Let N_1 and N_2 be two lattices in the same Carnot group G . Let ϕ_1 and ϕ_2 be injective, nonsurjective endomorphisms of N_1 and N_2 respectively, each acting without unipotent part. Suppose that there are integers r_1, r_2 such that $M_1^{r_1}$ and $M_2^{r_2}$ have the same permuted absolute Jordan form. Then Γ_{N_1, ϕ_1} and Γ_{N_2, ϕ_2} are quasi-isometric.*

In the case of abelian-by-cyclic groups, absolute Jordan form is a quasi-isometry invariant. The absolute Jordan form captures very specific information about a matrix. It is determined by the absolute values of the eigenvalues and the dimensions of the corresponding root spaces. This information determines quasi-isometry type by identifying the rates at which vectors grow when repeatedly multiplied by the matrix M . A pure eigenvector grows as λ^t , where λ is the corresponding eigenvalue. A vector in the λ -root space grows as $t^n \cdot \lambda^t$ for some $n \in \mathbb{N}$. They proved that these growth rates are quasi-isometry invariants of the abelian-by-cyclic group and that these rates uniquely determine the absolute Jordan form of M .

In the case of nilpotent-by-cyclic groups, similarly defined divergence rates are still quasi-isometry invariants. For each $x \in N$, consider the function $f_x(t) = d(0, \phi^t(x))$. After defining a suitable equivalence relation for functions from \mathbb{R} to \mathbb{R} , we consider the set of all divergence rates, up to this equivalence relation:

$$\mathcal{D}_\phi = \{[f_x(t)] \mid x \in N\}.$$

See Section 10 for precise definitions.

Theorem 3 (Divergence Rates are a Quasi-isometry Invariant). *Let N_1 and N_2 be lattices in the same Carnot group G . Let ϕ_1 and ϕ_2 be injective, nonsurjective endomorphisms of N_1 and N_2 respectively, each acting without unipotent part. If Γ_{N_1, ϕ_1} and Γ_{N_2, ϕ_2} are quasi-isometric then the multisets of divergence rates \mathcal{D}_{ϕ_1} and \mathcal{D}_{ϕ_2} are equal.*

Because the nilpotent group has interesting geometry of its own, the set of divergence rates is not sufficient to specify the absolute Jordan form. We use the permuted absolute Jordan form in order to keep some information about the geometry of the nilpotent group, and this further information ensures that the divergence rates can be calculated from the permuted absolute Jordan form. Nonetheless, the permuted absolute Jordan form is not uniquely determined by

the set of divergence rates, so, unlike the abelian-by-cyclic case, this does not establish permuted absolute Jordan form as a quasi-isometry invariant.

The set of divergence rates fails to determine the permuted absolute Jordan form in part because it fails to capture which divergence rates arise from points in the various levels of the nilpotent group. Part of this data is found by considering the Lie subalgebras associated to various growth rates. We consider the *growth spaces*:

$$\mathfrak{g}_\lambda = \{v \in \mathfrak{g} \mid \|M^t v\| \preceq \lambda^t t^k \text{ for some } k \in \mathbb{N}\}.$$

(See Section 18 for precise definitions.) Pansu has shown [P] that quasi-isometry Carnot groups have isomorphic Lie algebras. This implies that the isomorphism type of the growth spaces is a quasi-isometry invariant of the nilpotent-by-cyclic groups.

Theorem 4 (Growth Spaces are a Quasi-isometry Invariant). *Let N_1 and N_2 be lattices in the same Carnot group G . Let ϕ_1 and ϕ_2 be injective, nonsurjective endomorphisms of N_1 and N_2 respectively, each acting without unipotent part. If Γ_{N_1, ϕ_1} and Γ_{N_2, ϕ_2} are quasi-isometric then the each growth space \mathfrak{g}_{λ_1} of Γ_{N_1, ϕ_1} is isomorphic (as a Lie algebra) to some growth space \mathfrak{g}_{λ_2} of Γ_{N_2, ϕ_2} .*

The remainder of this chapter will contain an outline of the classification results. The second chapter focuses on a single nilpotent-by-cyclic group, first describing precisely how the permuted absolute Jordan form is computed and then showing how that data determines the geometry of a particular associated Lie group. In Chapter 10, a geometric model space for the nilpotent-by-cyclic groups is described, followed by proofs of Theorems 2 - 4. Some low-dimensional examples of nilpotent-by-cyclic groups are described in Chapter 18. A proof of the rigidity result follows in the last chapter.

2 Outline of the Classification

Step 1: (Section 11) Given a nilpotent-by-cyclic group $\Gamma = N_\phi$, where N is a finitely-generated nilpotent group and ϕ is an injective endomorphism of N , we construct a geometric model space X which is quasi-isometric to Γ . Topologically $X = \mathbb{R}^n \times T$, where T is the Bass-Serre tree for Γ . Metrically, for each $x \in T$, the horizontal slice $\mathbb{R}^n \times \{x\}$ has the metric given by the pullback of the metric on $\mathbb{R}^n \times \{0\}$ via $(\phi^t)^*$, where $t = \text{height of } x$. This construction follows the same process as that used in [FM00], but the metric on each slice is the non-isotropic geometry of nilpotent groups which is described in Section 3.

Step 2: (Section 13) We extend the results used by Farb and Mosher in [FM00] to show that the group defined by a given matrix is quasi-isometric to the group defined by its absolute Jordan form. This requires showing that the conjugating matrices preserve the nilpotent structure of the base group.

Step 3: (Section 14) Suppose f is a quasi-isometry between two nilpotent-by-cyclic groups $\Gamma = \Gamma_{N,\phi}$ and $\Gamma' = \Gamma_{N',\phi'}$. With the additional condition that ϕ and ϕ' are not surjective, we apply the same coarse topology as in [FM00] to get a quasi-isometry between the Lie groups which are hyperplanes in the model spaces: $G = N \rtimes_{\phi} \mathbb{R}$ and $G' = N' \rtimes_{\phi'} \mathbb{R}$. This result depends upon the groups Γ and Γ' being nonpolycyclic, which holds when the Bass-Serre trees $T_{\phi}, T_{\phi'}$ have valence $v \geq 3$. These hyperplanes correspond to $\mathbb{R}^n \times l$ for some directed line $l \subset T$. As a result, we show that if two nilpotent-by-cyclic groups are quasi-isometric then their base nilpotent groups must also be quasi-isometric. Having reduced this part of the problem to the (unsolved) quasi-isometric classification of nilpotent groups, we thenceforth restrict our attention to nilpotent-by-cyclic groups which are HNN extensions of two nilpotent groups which are lattices in the same nilpotent Carnot group.

Step 4: (Sections 16 - 17) The quasi-isometry between the hyperplanes has even more structure. Under the additional condition that ϕ and ϕ' act without unipotent part, we show that vertical flow lines are coarsely preserved. For any nilpotent-by-cyclic group, we calculate the divergence rates for pairs of vertical flow lines and show that the set of such rates is a finite set which is determined by the permuted absolute Jordan form (Sections 8 - 10). By considering the form of such divergence rates, we show that the time change function induced by the quasi-isometry is linear, so the set of rates at which vertical lines diverge from the flow line at the origin is a quasi-isometry invariant, up to rescaling all the rates by a single power.

Step 5: (Section 18) We use the results of Chapter 2 on divergence rates to show that the growth spaces are Lie subalgebras and then apply Pansu's theorem to establish Theorem 4.

Calculating the Quasi-isometry Invariants

The main result of this chapter is contained in Theorem 16, which establishes the set of “divergence rates” in Lie groups associated to nilpotent-by-cyclic groups. These rates are one of the two quasi-isometry invariants established in Chapter 10. We also establish, in Section 6, that two groups with the same permuted absolute Jordan form are quasi-isometric.

In the early sections of this chapter, we present some tools which will be needed for the proof in Section 10. In Section 3, we describe the geometry of left-invariant metrics on nilpotent groups. We establish some characteristics of endomorphisms of torsion-free nilpotent groups in Section 4, and describe a needed assumption on the endomorphism in Section 5. The definition of permuted absolute Jordan form is found in Section 6. Section 7 contains a key result from linear algebra which relates the structure of a nilpotent Lie group to the structure of any endomorphism of its Lie algebra. Section 12 describes and strengthens a result on the relationship between a matrix and its absolute Jordan form, in preparation for the proof of Theorem 2 in the following section. In Section 8, we introduce an equivalence relation which allows us to distinguish between different divergence rates. Section 9 expands upon results of [FM00]

to describe the growth of a vector under repeated application of a linear map. Finally, all these pieces are used in the final section to establish the set of divergence rates for a nilpotent-by-cyclic group.

3 Geometry of Nilpotent Groups

We consider the class of left-invariant Riemannian metrics on connected, simply-connected nilpotent Lie groups. As for discrete groups, any two left-invariant metrics on the same Lie group are quasi-isometric, so the quasi-isometric classification of groups, independent of the particular metric, is a well-defined problem.

Such groups can be globally coordinatized by \mathbb{R}^n such that balls centered at the origin are comparable to ellipsoids with axes of length which is polynomial in the radius, relative to the coordinates of \mathbb{R}^n .

Nilpotent groups admit both Riemannian metrics and so-called Carnot-Caratheodory (CC) metrics. Typical CC metrics are non-Riemannian on the infinitesimal scale and are non-isotropic on the large scale. The Riemannian metrics of interest to us are trivially CC metrics. They are infinitesimally Euclidean, unlike typical CC metrics. However, on the large-scale, Riemannian metrics and CC metrics are similarly non-isotropic. This non-isotropic nature is revealed by the description of balls in this metric, which is found below in Theorem 5.

Given a finitely-generated nilpotent group N , we define $\gamma_1(N) = N$, $\gamma_2(N) = [\gamma_1(N), \gamma_1(N)]$, and inductively, $\gamma_{i+1}(N) = [\gamma_1(N), \gamma_i(N)]$. Then

$$N = \gamma_1(N) \supset \cdots \supset \gamma_{c+1}(N) = 1$$

is the lower central series of N . We will refer to this filtration on N as the *nilpotent grading*. Define:

$$d_i = \dim(\gamma_i / \gamma_{i+1}).$$

We choose a basis for the Lie algebra which respects this grading:

Definition (Triangular basis). Suppose $\{e_1, \dots, e_n\}$ is a basis for the nilpotent Lie algebra \mathfrak{n} such that $[e_i, e_j] = \sum_k \alpha_{ijk} e_k$. The basis is triangular if $\alpha_{ijk} = 0$ when $k \leq \max(i, j)$. The constants α_{ijk} are called the structure constants for the group.

Example. The vectors $\{X, Y, Z\}$ form a triangular basis for the Lie algebra of the Heisenberg group, because $[X, Y] = Z$, $[X, Z] = 0$, and $[Y, Z] = 0$. The basis $\{X + Z, Y, Z\}$ is also triangular, but neither $\{X, Y, X + Z\}$, nor the reordered basis $\{Z, Y, X\}$ is.

Each vector $v \in \mathfrak{n}$ is assigned a weight $w(v)$ which specifies the last component of the nilpotent grading which contains v :

$$w(v) = \max\{i \mid e_k \in \gamma_i(\mathfrak{g})\}.$$

For a given choice of (ordered) basis, we will call the associated n -tuple (w_1, \dots, w_n) the *weight vector* associated to the basis.

Example. In the Heisenberg group, given in $\{X, Y, Z\}$ coordinates, the weights are: $w(X) = w(Y) = 1; w(Z) = 2$. That is, the weight vector is $(1, 1, 2)$.

In this notation, both Gromov ([G96], for Carnot-Caratheodory spaces) and Karidi have shown ([K], Theorem 4.2):

Theorem 5 (Ball-Box Comparison Theorem). *Let N be a connected, simply-connected, real, nilpotent Lie group of dimension n with Lie algebra \mathfrak{n} , and let $\{e_1, \dots, e_n\}$ be a triangular basis of \mathfrak{n} with associated weight vector $w = (w_1, \dots, w_n)$. Specifying $\{e_1, \dots, e_n\}$ as an orthonormal basis defines an inner product at the origin. Applying left-invariance, this defines a metric on the group N . Let $B(r)$ be the ball centered at the origin in N with radius $r > 1$ in this metric.*

Then there exists a constant $a > 1$ (which depends on the group N , but not on r) such that

$$\{|x_i| \leq (r/a)^{w_i} \mid i = 1, \dots, n\} \subset B(r) \subset \{|x_i| \leq (ar)^{w_i} \mid i = 1, \dots, n\}.$$

Definition. Two functions $a, b: X \rightarrow \mathbb{R}$ are comparable, denoted $a(x) \sim b(x)$ if there exists $K > 0$ such that for all $x \in X$

$$\frac{1}{K}b(x) < a(x) < Kb(x).$$

Example. In the Heisenberg group, given in $\{X, Y, Z\}$ coordinates, the ball of radius r is comparable to the box of the form $[-r, r] \times [-r, r] \times [-r^2, r^2]$.

Karidi's result on balls can be rephrased as follows to describe the distance between points.

Corollary 6 (Distances in Nilpotent Groups). *For a nilpotent Lie group N , with weights w_i as in Theorem 5, the left-invariant metric:*

$$\|(x_1, \dots, x_n)\| \sim \max_i \{|x_i|^{\frac{1}{w_i}}\}.$$

Proof. For $x = (x_1, \dots, x_n)$, let $d = \max_i \{|x_i|^{\frac{1}{w_i}}\}$. Then each $|x_i|^{\frac{1}{w_i}} \leq d$, so $|x_i| \leq d^{w_i}$. By Theorem 5, the box $\{|x_i| \leq d^{w_i}\} \subset B(ad)$, so $x \in B(ad)$.

Conversely, there is some i such that $|x_i|^{\frac{1}{w_i}} = d$, so $|x_i| = d^{w_i}$. Again, Theorem 5 implies that for all $\epsilon > 0$, $x \notin \{|x_i| \leq (d - \epsilon)^{w_i}\}$, which contains $B(\frac{d - \epsilon}{a})$. Thus, $\frac{d - \epsilon}{a} < \|x\| < ad$ for all $\epsilon > 0$, so $\frac{d}{a} \leq \|x\| < ad$.

$$\|(x_1, \dots, x_n)\| \sim \max_i \{|x_i|^{\frac{1}{w_i}}\}.$$

□

At times, it will be convenient to use the following characterization of this distance metric:

Corollary 7 (Distances in Nilpotent Groups, Part 2). *Given a triangular basis $\{e_i\}$ with weights $\{w_i\}$, consider*

$$V_j = \text{span}\{e_i \mid w_i = j\}.$$

Then for some k , $G \cong R^n = V_1 \oplus \cdots \oplus V_k$. Given $x \in G$, we express x uniquely as $x = (x_1, \dots, x_k)$, with each $x_i \in V_i$. Then,

$$\|x\| \sim \max_i \{\sqrt[i]{|x_i|}\}.$$

Proof. This follows immediately from the definitions and Corollary 6. \square

Example. In the Heisenberg group, the distance from the origin to the point (a, b, c) (which represents the group element $x^a y^b z^c$) is comparable to the function $\max\{a, b, \sqrt{c}\}$.

The following four-step nilpotent group will be used as the basis for examples of nilpotent-by-cyclic groups in Section 20.

Example. Define

$$\begin{aligned} G = \langle x, y, z, a, b, c, p, q, r, s, t \mid & [x, y] = z, [a, b] = c, [z, c] = t, \\ & [p, q] = r, [p, r] = s, [q, r] = s, \\ & [r, r] = t, [p, s] = t, [q, s] = t \rangle. \end{aligned} \quad (1)$$

Notice that (x, y, z) , (a, b, c) , (p, q, r) and (z, c, t) are each isomorphic to the Heisenberg group H . This nilpotent group has:

$$\begin{aligned} V_1 &= \langle x, y, a, b, p, q \rangle \\ V_2 &= \langle z, c, r \rangle \\ V_3 &= \langle s \rangle \\ V_4 &= \langle t \rangle \end{aligned}$$

Thus,

$$d(0, (x, y, z, a, b, c, t)) \simeq \max\{x, y, a, b, p, q, \sqrt{z}, \sqrt{c}, \sqrt{r}, \sqrt[3]{s}, \sqrt[4]{t}\}.$$

4 Some Characteristics of Endomorphisms of Nilpotent Groups

Recall that a nilpotent-by-cyclic group can be expressed as the HNN extension of a finitely-generated nilpotent group N by an injective endomorphism ϕ . We will now consider the endomorphisms of nilpotent groups which can define such an HNN extension. Their classification is an open question and is not considered here. However, we will establish a few important facts about such endomorphisms.

Malcev has shown ([M], Theorem 5) that an automorphism of a discrete nilpotent group Γ can be extended to an automorphism of any nilpotent Lie group in which Γ is a lattice. This result has been broadly generalized, for example by Raghunathan ([R], Theorem 2.11, p. 33). We will use the following statement:

Theorem 8. *Given an injective endomorphism ϕ of the discrete nilpotent group Γ , we can extend to an injective endomorphism $\hat{\phi}$ of the nilpotent Lie group G in which Γ is a lattice. As always, this extends to an automorphism of the Lie algebra \mathfrak{g} .*

The algebraic structure of a nilpotent group places significant restrictions on the structure of its endomorphisms. The restrictions are most easily described for a special class of nilpotent group known as *Carnot groups*, defined below. As it turns out, these groups are the ones which admit the largest classes of endomorphisms.

Definition. *A connected, simply-connected nilpotent Lie group G is a Carnot group if its Lie algebra \mathfrak{g} admits a grading $\mathfrak{g} = \bigoplus_{j=1}^r V_j$ such that:*

- $[V_1, V_j] = V_{j+1}$ for $1 \leq j < r$, and $[V_1, V_r] = \{0\}$, and
- V_1 generates all of \mathfrak{g} via $[\cdot, \cdot]$.

Any nilpotent Lie group has a naturally associated Carnot group, which is obtained by redefining each commutator, omitting any term which is in a higher grade than permitted by the Carnot definition.

Definition. *Given a linear map ϕ of a vector space V , and a grading $V = \bigoplus_{j=i}^r V_j$, we say that ϕ weakly preserves the grading of V if $\phi(\bigoplus_{j=i}^r V_j) \subseteq (\bigoplus_{j=i}^r V_j)$ for each $i = 1, \dots, r$.*

Lemma 9. *Let G be a Carnot group, and let ϕ be an injective endomorphism of the Lie algebra \mathfrak{g} . Then, ϕ weakly preserves the Carnot grading $\mathfrak{g} = \bigoplus_{j=1}^r V_j$.*

Proof. Proof is by induction. The base case, that $\phi(\bigoplus_{j=1}^r V_j) \subseteq \bigoplus_{j=1}^r V_j$ is trivially satisfied.

Now, assume that $\phi(\bigoplus_{j=i}^r V_j) \subseteq (\bigoplus_{j=i}^r V_j)$ for each $i = 1, \dots, k$. Each $v \in \bigoplus_{j=k+1}^r V_j$ can be expressed as a commutator $v = [x, y]$ for some $x \in \mathbb{R}^n, y \in \bigoplus_{j=k}^r V_j$. Then, $\phi(v) = [\phi(x), \phi(y)]$, with $\phi(x) \in \mathfrak{g}$ and $\phi(y) \in \bigoplus_{j=k}^r V_j$, so $\phi(v) \in \bigoplus_{j=k+1}^r V_j$. \square

The structure of the nilpotent group does give even more restrictions on the endomorphism. In particular, for a Carnot group, the action of ϕ on the base level, V_1 , determines completely the action on the rest of the group.

Example. An endomorphism of the Heisenberg group can be described via the 3×3 matrix which represents the endomorphism for the basis $\{x, y, z\}$. Given the first two columns (representing the effect of ϕ on x and y) the third column (representing the effect of ϕ on z) can be calculated explicitly, since $\phi(z) = [\phi(x), \phi(y)]$. If $\phi(x) = x^a y^c z^e$ and $\phi(y) = x^b y^d z^f$, then $\phi(z) = z^{\det}$, where $\det = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$, the determinant of the matrix representing $\phi|_{\langle x, y \rangle}$. So, a generic matrix representing an endomorphism of the Heisenberg group has the form:
$$\begin{bmatrix} a & b & 0 \\ c & d & 0 \\ e & f & \det \end{bmatrix}.$$

5 Unipotent-Free Endomorphisms

Definition. An endomorphism ϕ of a nilpotent group G is called unipotent-free if the matrix representing ϕ has no unipotent part, that is, no Jordan block with eigenvalue on the unit circle.

We now make a:

Standing Assumption: The endomorphism ϕ of G is unipotent-free.

This assumption was not necessary in the abelian-by-cyclic case. The extension of an abelian group by unipotent matrix U is a nilpotent group of the form $\mathbb{Z}^k \rtimes_U \mathbb{Z}$. As shown by Bridson and Gersten [BG], Pansu's invariant [P] shows that the sizes of the unipotent blocks in U is a quasi-isometry invariant for such groups. This yields a quasi-isometric classification ([FM00], Corollary 5.6) of these components of the abelian-by-cyclic groups.

In contrast, excluding endomorphisms with unipotent parts is essential to the classification given here. In fact, all nilpotent groups can be expressed as the HNN extension of a simpler nilpotent group by a unipotent matrix. Thus, to classify nilpotent-by-cyclic groups defined with unipotent matrices would be a very big step in the classification of nilpotent groups. It seems that we could use Pansu's invariants from [P] as in [FM00] to any unipotent part which acts on the center $Z(G)$, but we will not carry out that work in this dissertation.

If an endomorphism fixes any vector, then it has unipotent part. Thus, any nilpotent-by-cyclic group which is an HNN extension defined by a unipotent-free endomorphism has trivial center. However, a centerless nilpotent-by-cyclic group may be defined by an endomorphism which is not unipotent-free if the unipotent part acts on elements of G which are not in the center of G .

6 Permuted Absolute Jordan Form

The well-known Jordan form theorem states that any square matrix with complex entries is conjugate to a matrix with the canonical Jordan form. In this section, we describe a modification of this form: *permuted absolute Jordan form*.

Let $\mathcal{M}_n(F)$ denote all $(n \times n)$ -matrices over a field F , and let $\text{GL}_n(F)$ be the group of invertible matrices.

Definition. An matrix $J_n(\lambda) \in \mathcal{M}_n(\mathbb{C})$ is a Jordan block with eigenvalue λ if $J_n(\lambda) = (a_{ij})$ with

$$a_{ij} = \begin{cases} \lambda & i = j \\ 1 & i = j - 1 \\ 0 & i \neq j - 1, j. \end{cases}$$

That is, $J_n(\lambda)$ has λ 's along the diagonal, 1's on the superdiagonal, and zeros elsewhere.

Definition. A matrix $J \in \mathcal{M}_n(\mathbb{R})$ is a real Jordan block if it has one of the following two forms. The first form is an ordinary Jordan block $J_n(\lambda)$ where $\lambda \in \mathbb{R}$. The second form, which requires n to be even, has a 2×2 block decomposition of the form

$$J = J_n(a, b) = \begin{pmatrix} Q(a, b) & Id & \dots & 0 & 0 \\ 0 & Q(a, b) & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & Q(a, b) & Id \\ 0 & 0 & \dots & 0 & Q(a, b) \end{pmatrix}$$

where Id is the identity, 0 is the zero matrix, $Q(a, b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, and $b \neq 0$.

Definition. Given a partition $N = n_1 + \dots + n_k$, and blocks J_i for $i = 1, \dots, k$, we define the matrix

$$M = (J_1, J_2, \dots, J_k) = \begin{pmatrix} J_1 & 0 & \dots & 0 \\ 0 & J_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & J_k \end{pmatrix}.$$

If the blocks J_i are all real Jordan blocks, then we say that matrix M is in real Jordan block form. If the blocks J_i are all Jordan blocks, then we say that matrix M is in complex Jordan block form. If the blocks J_i are all Jordan blocks, of the first type (that is, Jordan blocks with real λ), then we say that M is in Jordan block form.

It is a standard result of linear algebra that every matrix $M \in \mathcal{M}_n(\mathbb{C})$ is conjugate via an element in $\text{GL}_n(\mathbb{C})$ to a matrix in complex Jordan form, which is unique up to permutation of the Jordan blocks. Similarly, every matrix $M \in \mathcal{M}_n(\mathbb{R})$ is conjugate via an element in $\text{GL}_n(\mathbb{R})$ to a matrix in real Jordan form, which is also unique up to permutation of the Jordan blocks.

Definition. The absolute Jordan form of $M \in \mathcal{M}_n(\mathbb{R})$ is the matrix obtained by replacing each diagonal entry of the complex Jordan form with its absolute value.

We will resolve the nonuniqueness in absolute Jordan form by specifying:

- if $|\lambda_i| > |\lambda_j|$, then $i > j$, and
- if $|\lambda_i| = |\lambda_j|$, and $n_i > n_j$, then $i > j$.

Definition. Suppose $<$ defines a partial order on the set $\mathbb{N}_n = 1, \dots, n$. We say a permutation $\sigma \in S_n$ preserves the partial order $<$ if

$$a < b \implies \sigma(a) < \sigma(b).$$

Definition. If a matrix $M = (J_{n_1}(\lambda_1), J_{n_2}(\lambda_2), \dots, J_{n_k}(\lambda_k))$ is in Jordan form, let $N_0 = 0$ and $N_i = n_1 + \dots + n_{i-1}$. Define a partial order $<_M$ on \mathbb{N}_n by: $i <_M j$ if:

- $i < j$ as natural numbers, and
- $N_{l-1} < i, j \leq N_l$ for some $l = 1, \dots, k$.

If there is no l such that $N_{l-1} < i, j \leq N_l$, then i and j are not comparable by $<_M$. We say $<_M$ is the partition associated to the Jordan matrix M .

Definition. A matrix $M = (m_{i,j})$ is in permuted Jordan form if there is a permutation $\sigma \in S_n$ such that

- the matrix $M_\sigma = (m_{\sigma(i), \sigma(j)})$ is in Jordan form, and
- the permutation σ^{-1} preserves the partial order $<_M$ associated to M_σ .

The diagonal entries of M_σ will be the diagonal elements of M , permuted by σ . As a consequence of the second condition of the definition, all the off-diagonal 1's will be above the diagonal of matrix M .

Given an endomorphism ϕ of a Carnot group G , we associate to this a canonical matrix in permuted absolute Jordan form as follows:

Definition. Consider an endomorphism ϕ of an n -dimensional nilpotent Lie group G . The induced linear map of the Lie algebra can be represented with respect to some basis $\{e_i\}$ by a matrix M in Jordan form. By Proposition 10 this basis respects the nilpotent grading. Therefore, Theorem 5 associates a set of integer weights $\{w_i\}$ to the basis $\{e_i\}$. We define a permutation $\sigma \in S_n$ so that:

- if $w_i > w_j$, then $\sigma(i) < \sigma(j)$, and
- if $w_i = w_j$, then $\sigma(i) < \sigma(j)$ if and only if $i < j$.

The effect of these requirements is that we do as little rearranging of the Jordan form as possible, subject to the condition that the new weight vector is non-increasing.

Example. Recall the three-step nilpotent group G described at the end of Section 3. Consider the endomorphism of the associated Lie group \mathfrak{g} which is represented by the matrix:

$$M = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix},$$

with associated weight vector $w = (1, 2, 3, 1, 1, 2)$.

Then the permuted absolute Jordan form will be obtained with the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 6 & 2 & 3 & 5 \end{pmatrix}.$$

The permuted absolute Jordan matrix will be:

$$M_\sigma = \begin{pmatrix} 2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}.$$

7 Linear Algebra

In order to understand how the induced map ϕ^* acts on \mathfrak{g} , we need the following results of linear algebra. Recall that ϕ weakly preserves a decomposition $V = L_1 \oplus \cdots \oplus L_k$ if $\phi(L_i \oplus \cdots \oplus L_k) \subset L_i \oplus \cdots \oplus L_k$ for all $i = 1, \dots, k$.

Proposition 10. *Let ϕ be a nonsingular linear map of a vector space V . Suppose that V has a direct sum decomposition $V = L_1 \oplus \cdots \oplus L_k$ which is weakly preserved by ϕ . Then there is a basis $\{e_i\}$ for V such that*

- a) *The matrix representing ϕ in the basis $\{e_1, \dots, e_n\}$ is in permuted Jordan form.*
- b) *The permutation which puts the matrix in permuted Jordan form respects the partial order associated to the Jordan form. (Equivalently, the matrix is in upper triangular form)*

c) If $d(i) = \dim(L_i \oplus \cdots \oplus L_k)$, then $L_i \oplus \cdots \oplus L_k = \text{span}\{e_1, \dots, e_{d(i)}\}$

The proof of the theorem will rely upon the following lemma, in which we consider the special case that ϕ acts as a single Jordan block.

Lemma 11. *Suppose ϕ is a linear map of an n -dimensional vector space V , and that ϕ can be represented as a matrix which is a single real Jordan block with respect to the basis $\{e_1, \dots, e_n\}$. Let $V_i = \text{span}\{e_1, \dots, e_i\}$. If $W \subset V$ is a subspace preserved by ϕ , (i.e., $\phi(W) \subset W$), then $W = V_{\dim(W)}$.*

Proof. Case 1: The real Jordan block representing ϕ is $J_n(\lambda)$.

Suppose $W \neq V_i$ for all i . Let i be maximized subject to the condition that $V_i \subset W$. Yet, by assumption, $W \neq V_i$, so we can choose some $w \in W \cap \text{span}\{e_{i+1}, \dots, e_n\}$, which can be expressed as $w = a_{i+1}e_{i+1} + \cdots + a_n e_n$. Let j be maximized subject to $a_j \neq 0$; that is, $w \in W \cap \text{span}\{e_{i+1}, \dots, e_j\}$. We will demonstrate that there is some $w' \in W \cap \text{span}\{e_{i+1}, \dots, e_{j-1}\}$.

This will imply, by induction, that there is some element $w^* \in W \cap \text{span}\{e_{i+1}\}$, which will imply that $V_{i+1} \subset W$. This contradiction will imply that $W = V_i$, for some i . Then, i is determined so that $\dim(W) = \dim(V_i) = i$.

To construct w' , note that $\phi(w) = \lambda w + a_2 e_1 + \cdots + a_i e_{i-1} + a_{i+1} e_i + \cdots + a_j e_{j-1}$. But $\lambda w \in W$ and $a_2 e_1 + \cdots + a_i e_{i-1} + a_{i+1} e_i \in V_i \subset W$, so $w' = a_{i+2} e_{i+1} + \cdots + a_j e_{j-1} \in W$ as well.

Case 2: The real Jordan block representing ϕ is $J_n(a, b)$ (and therefore $n = 2m$ is even).

This case proceeds similarly to the first, except that elements in the vectors are replaced by ordered pairs. Given an element $w \in \mathbb{R}^n$, express $w = ((c_1, d_1), \dots, (c_m, d_m))$, where each $(c_i, d_i) \in \mathbb{R}^2$.

Again, assume that $W \neq V_i$ for all i . Let i be maximized subject to the condition that $V_{2i} \subset W$. Yet, by assumption, $W \neq V_{2i}$, so we can choose some $w \in W \cap \text{span}\{e_{i+1}, \dots, e_n\}$, which can be expressed as $w = c_{i+1} e_{2i+1} + d_{i+1} e_{2i+2} + \cdots + c_m e_{n-1} + d_m e_n$. Let j be maximized subject to $(c_j, d_j) \neq (0, 0)$; that is, $w \in W \cap \text{span}\{e_{2i+1}, \dots, e_{2j}\}$. We will demonstrate that there is some $w' \in W \cap \text{span}\{e_{2i+1}, \dots, e_{2j-2}\}$.

But, by induction, this implies that there is some element $w^* \in W \cap \text{span}\{e_{i+1}, e_{i+2}\}$. Furthermore, $\phi(w^*) \in W \cap \text{span}\{e_{i-1}, e_i, e_{i+1}, e_{i+2}\}$. Since $\text{span}\{e_{i-1}, e_i\} \subset W$, we can subtract the e_{i-1} and e_i components from $\phi(w^*)$. The result is an element of $\text{span}\{e_{i+1}, e_{i+2}\}$ which is linearly independent of w^* . Since w^* , this implies that $V_{i+2} \subset W$. This contradiction will imply that $W = V_i$, for some i . Then, i is determined so that $\dim(W) = \dim(V_i) = i$.

Now we construct w' . Let $Q = Q(a, b)$, so that

$$\phi(w) = (Q(c_1, d_1) + (c_2, d_2), Q(c_2, d_2) + (c_3, d_3), \dots, Q(c_m, d_m)).$$

Since (c_m, d_m) and $Q(c_m, d_m)$ are linearly independent vectors in \mathbb{R}^2 , there are constants $r, s \in \mathbb{R}$ such that $r(c_m, d_m) + sQ(c_m, d_m) = 0$. Thus, $rw + s\phi(w) \in W \cap \text{span}\{e_1, \dots, e_{2j-2}\}$. Since $V_{2i} \subset W$, we can subtract off the necessary terms of $rw + s\phi(w)$ to obtain $w^* \in W \cap \text{span}\{e_{i+1}, \dots, e_{2j-2}\}$. \square

Proof of Proposition 10. It is a standard result of linear algebra (Jordan Decomposition Theorem) that every nonsingular linear map on a vector space can be decomposed into the action on a direct sum of ϕ -invariant root spaces, such that the induced action on each root space is as a single real Jordan block.

Let the root space decomposition of V induced by ϕ be $V = R_1 \oplus \cdots \oplus R_m$. On each R_i , the map ϕ can be represented as a single real Jordan block with respect to some basis for R_i , say, $\{a_1^i, \dots, a_{\dim(R_i)}^i\}$. Let $V_j^i = R_i \cap (L_j \oplus \cdots \oplus L_k)$. Since ϕ preserves both R_i and $L_j \oplus \cdots \oplus L_k$, it also preserves V_j^i . Thus, we apply Lemma 9 to prove that

$$V_j^i = \text{span}\{a_1^i, \dots, a_{\dim(V_j^i)}^i\}.$$

Define a function $f: \{a_j^i\} \rightarrow \{1, \dots, n\}$ by listing the a_j^i as follows:

$$a_1^1, \dots, a_{\dim V_1^1}^1, a_1^2, \dots, a_{\dim V_1^2}^2, \dots, a_1^m, \dots, a_{\dim V_1^m}^m, \dots$$

$$a_{\dim V_1^1+1}^1, \dots, a_{\dim V_2^1}^1, \dots, a_{\dim V_k^m}^m.$$

Then, f maps the i^{th} element of this list to i . Thus, the first $\dim V_1^1 + \cdots + \dim V_1^m = \dim L_1$ vectors span L_1 , and similarly for L_j . Also, for fixed i , the order of $\{a_j^i\}$ is preserved. Thus, the matrix representing ϕ with respect to the basis $\{e_i\}$ is in permuted Jordan form. Furthermore, this permutation respected the partial order associated to the Jordan matrix. \square

8 An Order on Divergence Rates

In order to compare the rates at which the lengths of different vectors grow, we will describe a partial order and an equivalence relation on functions from \mathbb{R} to \mathbb{R} . The order will characterize functions which dominate others.

Definition. Given $f, g: \mathbb{R} \rightarrow \mathbb{R}$ we say that $f \preceq g$ if there are $K, C \geq 0$ such that

$$f(t) \leq Kg(t) + C \text{ for all } t \in \mathbb{R}.$$

We say f, g are comparable ($f \simeq g$) if $f \preceq g$ and $g \preceq f$.

Equivalently, $f \simeq g$ if there exist constants $K', C' > 0$ such that:

$$\frac{1}{K'}g(t) - C' \leq f(t) \leq K'g(t) + C' \text{ for all } t \in \mathbb{R}.$$

This is an equivalence relation:

- (Symmetry) For all $f, f \simeq f$ with constants $K = 1, C = 0$
- (Transitivity) If $f \simeq g$ with constants K and C , and $g \simeq h$ with constants K' and C' , then $f \simeq h$ for constants KK' and $\max\{K'C + C, \frac{C}{K'} + C'\}$

- (Reflexivity) If $f \simeq g$ with constants K and C , then $g \simeq f$ with constants K and $\max\{CK, \frac{C}{K}\}$.

We denote the equivalence class of a function f by $[f]$. In the abelian-by-cyclic case, the corresponding divergence rates are always exponential or polynomial-exponential functions. In Theorem 16, we will show that divergence rates are exponential, polynomial-exponential, or roots thereof. For this class of functions, the partial order is actually an order.

Lemma 12. *For the class of functions $\mathcal{C} = \{f(t) = (t^n \cdot \lambda^t)^{\frac{1}{d}} \mid \lambda \neq 1, \lambda \geq 0, n, d, \in \mathbb{Z}^+\}$, the partial order \preceq is an order.*

Proof. Consider two such functions $f(t) = (t^n \cdot \lambda^t)^{\frac{1}{d}}$ and $g(t) = (t^m \cdot \kappa^t)^{\frac{1}{c}}$ with $\lambda, \kappa > 0, n, m, c, d \in \mathbb{Z}^+$.

Case 1: $\lambda^{\frac{1}{d}} \neq \kappa^{\frac{1}{c}}$

In this case, the exponential growth dominates. Assume, without loss of generality, that $\lambda^{\frac{1}{d}} > \kappa^{\frac{1}{c}}$. If we set $C = 0$, we are looking for a value of K such that

$$\frac{(t^m \cdot \kappa^t)^{\frac{1}{c}}}{(t^n \cdot \lambda^t)^{\frac{1}{d}}} \leq K \text{ for all } t.$$

Elementary calculus shows that $f(t) = t^a b^t$ has a global maximum at $t = \frac{-a}{\ln(b)}$, where $a = \frac{m}{c} - \frac{n}{d}$ and $b = \frac{\kappa^{\frac{1}{c}}}{\lambda^{\frac{1}{d}}}$. Thus, choosing for K the maximal value of f shows that $g \preceq f$.

Case 2: $\lambda^{\frac{1}{d}} = \kappa^{\frac{1}{c}}$

In this case, the exponential parts of the functions grow at the same rate, so the degree of polynomial determines which function is larger. Assume, without loss of generality, that $\frac{n}{d} > \frac{m}{c}$. For all $t \geq 1$, $f(t) \geq g(t)$. Since $g(t) < \lambda$ for all $0 < t < 1$, we choose $C = \lambda$, $K = 1$, to show that $g \preceq f$. \square

We will also need the following Lemma.

Lemma 13. *Suppose $\{f_1, \dots, f_n\} \subset \mathcal{C}$ has $f_1 \preceq f_i$ for $i = 2, \dots, n$. Then $[\Sigma_{i=1}^n f_i] = [f_1]$.*

Proof. If $f_1 \preceq f_i$ with constants K_i, C_i , then $[\Sigma_{i=1}^n f_i] = [f_1]$ with constants $K_1 + \dots + K_n$ and $C_1 + \dots + C_n$. \square

9 Growth Rates of Vectors

Given a 1-parameter subgroup M^t of $\text{GL}_n(\mathbb{R})$, Farb and Mosher derive [FM00] upper and lower bounds for the growth of vectors $\|M^t v\|$. We will need more: bounds on the growth of each coordinate of $M^t v$. In the special case of $v = e_i$, one of the standard basis vectors for \mathbb{R}^n , results are actually contained in their proof and are stated below in Proposition 14. In Theorem 15 we extend to a result for arbitrary $v \in \mathbb{R}^n$. We will not give the growth function explicitly, but only up to the relation \simeq defined in Section 8.

Proposition 14. *Consider the Jordan block $M = J_n(\lambda) \in \text{GL}_n(\mathbb{R})$ (In particular, $\lambda \in \mathbb{R}$.) Let $\{e_i\}$ be the standard basis for \mathbb{R}^n . Then, considered as functions of t :*

$$e_k \cdot M^t \cdot e_j \simeq \begin{cases} \lambda^t t^{k-j} & j \leq k \leq n \\ 0 & k < j. \end{cases}$$

Proof. See [FM00], Equation 3.1. □

Now we extend this result to an arbitrary $x \in \mathbb{R}^n$:

Proposition 15. *Consider $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $M \in \text{GL}_n(\mathbb{R})$ a Jordan matrix. As functions of t :*

$$e_k \cdot M^t \cdot x \simeq \max_i \{e_k \cdot M^t \cdot e_i | x_i \neq 0\}.$$

Although the equivalence class of $e_k \cdot M^t \cdot x$ is independent of the nonzero values of x_i , the constants implicit in the relation \simeq does depend linearly on x_i for some i .

Proof. We write $x \in \mathbb{R}^n$ as $x = \sum_{i=1}^n x_i e_i$. Then,

$$M^t \cdot x = M^t \cdot \sum_{i=1}^n x_i e_i = \sum_{i=1}^n x_i M^t e_i.$$

Restricting our attention to the k^{th} coordinate:

$$e_k \cdot M^t \cdot x = \sum_{i=1}^n x_i e_k M^t e_i.$$

Now, we consider the divergence rate (as a function of t) of this coordinate, using the equivalence relation described in Section 8. Proposition 14 tells us that each of the component functions $e_k M^t e_i$ is in \mathcal{C} , and so, by Lemma 12, all are comparable. Lemma 13 implies that the sum of such functions is equivalent to the maximum. For fixed $x_i \neq 0$, we have $x_i f(t) \simeq f(t)$. Thus,

$$e_k \cdot M^t \cdot x \simeq \max_{i=1, \dots, n} \{e_k \cdot M^t \cdot e_i \mid x_i \neq 0\}.$$

□

10 Putting the Pieces Together: Divergence Rates

Given a connected, simply-connected nilpotent Lie group G and an injective endomorphism ϕ of G , we get an induced linear map ϕ^* on the Lie algebra \mathfrak{g} . Assume this can be represented by a Jordan matrix $M = (J_{n_1}(\lambda_1), J_{n_2}(\lambda_2), \dots, J_{n_k}(\lambda_k))$ with respect to some basis $\{e_i\}$. Let $N_i = n_1 + \dots + n_i$. By Lemma 9, this basis is also consistent with the nilpotent grading of \mathfrak{g} (although perhaps in a permuted order).

We define a Lie group $G_\phi = G \rtimes_\phi \mathbb{R}$, where the action of \mathbb{R} is given by ϕ^t . We identify G_ϕ with $\mathbb{R}^n \times \mathbb{R}$. Given any left-invariant metric on G , the metric on each slice $G_t = G \times \{t\}$ is given by the pullback of $\phi^t: G \rightarrow G$. See Section 11 for a description of the group operation.

In this section, we will calculate the rate of divergence of “vertical flow lines” in G_ϕ ; that is, for a given $x \in \mathbb{R}^n$, we calculate the distance from $(0, t)$ to (x, t) within G_t , which we denote by $f_x(t)$.

Recall that a multiset is similar to a set in that the elements do not have a designated order; it differs from a set in that elements may be repeated.

Definition. Given a Lie group G_ϕ as above, and a basis $\{e_i\}$ so that ϕ^* is represented as a real Jordan matrix, let $f_i(t) = d_t((0, t), (e_i, t))$. Recall that $[f]$ denotes the equivalence class of $f: \mathbb{R} \rightarrow \mathbb{R}$ as defined in Section 8. We define a multiset of divergence rates:

$$\mathcal{D}_\phi = \{[f_i(t)]\}.$$

Theorem 16. The multiset of divergence rates for a Lie group G_ϕ is:

- a) independent of generating set $\{e_i\}$;
- b) a finite subset of \mathcal{C} ; and
- c) contains at least one element of the form $[\lambda^t]$.

The proof of this theorem will require two lemmas. As in the abelian-by-cyclic case, we will see that $f_x(t) = \|M^{-t}(x)\|$, although this requires more work. On the other hand, now the norm $\|\cdot\|$ is the norm in the nilpotent metric, which depends on the grading of the nilpotent group, as discussed in Section 3:

$$\|(x_1, x_2, \dots, x_n)\| \sim \max_i \{x_i^{\frac{1}{w_i}}\},$$

where w_i = weight of x_i in the grading of N .

Lemma 17. With G_ϕ , $\{e_i\}$, and $f_i(t)$ given as above, reorder the $\{e_i\}$ so that

$$f_1(t) \preceq f_2(t) \preceq \dots \preceq f_n(t),$$

and define $V_i = \text{span}\{e_1, \dots, e_i\}$. If $x \in V_i \setminus V_{i-1}$, then $[f_x(t)] = [f_i(t)]$.

Notice that, although we write x as a vector, it is not a vector in the Lie algebra, but, rather, a fixed point in \mathbb{R}^n , which corresponds to a (varying) point in the Lie group, expressed in the coordinates corresponding to our chosen basis for the Lie algebra.

Proof. Let $\{e_i\}$ be an orthonormal basis for the metric on $G_0 = G \times \{0\}$. Then the metric on G_t can be defined equivalently by the orthonormal basis $e_i(t) = (\phi^t)^*(e_i) = M^t \cdot e_i$. The results described in Section 3 apply to each slice G_t to show that balls in this left-invariant metric are comparable to polynomial ellipsoids. The proof in [K] shows more. Because the commutivity data for

$\{e_i(t)\}$ is independent of t , the weights w_i and constant a in Theorem 5 are the same for each slice G_t . Thus, it suffices to write x in terms of this basis; i.e., $x = \sum_i x_i(t)e_i(t)$. This is simply a change of basis, so the coordinates $(x_1(t), \dots, x_n(t))$ are given by the vector $M^{-t} \cdot x$. That is,

$$d_t((0, t), (x, t)) = \|M^{-t}x\|.$$

By Corollary 6,

$$\|M^{-t}x\| = \max_k \{|e_k \cdot M^{-t} \cdot x|^{\frac{1}{w_k}}\}.$$

Applying Proposition 15,

$$\begin{aligned} \|M^{-t}x\| &= \max_k \max_i \{|e_k \cdot M^{-t} \cdot e_i|^{\frac{1}{w_k}} \mid x_i \neq 0\} \\ &= \max_i \{f_i(t) \mid x_i \neq 0\} \\ &= f_{\max\{i \mid x_i \neq 0\}}(t). \end{aligned} \tag{2}$$

Thus, for $x \in V_i \setminus V_{i-1}$, we have shown that $\|M^{-t}x\| \simeq f_i(t)$. \square

Proof of Theorem 16(a). Suppose that for a different Jordan basis $\{e'_i\}$, we obtain a different multiset of divergence rates:

$$g_1(t) \preceq g_2(t) \preceq \dots \preceq g_n(t).$$

Let i be maximized subject to $g_i \not\preceq f_i$ and assume, without loss of generality, that $g_i \prec f_i$. Let $V_i = \{x \in G \mid f_x \preceq g_i\}$. Then, $V_i \supset \text{span}\{e'_1, \dots, e'_i\}$ and so $\dim(V_i) \geq i$. On the other hand, $\text{span}\{e_i, \dots, e_n\} \cap V_i = \emptyset$, so $\dim V_i < i$. This contradiction shows that $f_i = g_i$ for all i . \square

Lemma 18. *With the notation as in Proposition 14, let $f_j(t) = f_{e_j}(t)$, and suppose e_k has weight w_k given by Theorem 5. Then,*

$$f_j(t) \simeq \max_k \{(t^{(k-j)} \lambda_i^t)^{\frac{1}{w_k}} \mid j \leq k \leq N_i\}.$$

Proof. We simply apply the metric determined in Theorem 5 to the coordinates determined by Proposition 14. \square

Proof of Theorem 16(b) and (c). Lemma 18 implies that each divergence rate $f_i \in \mathcal{C}$, which is (b).

For $j = N_i$ (corresponding to e_j is an eigenvector) we have $f_j(t) = (\lambda_j^{\frac{1}{w_j}})^t$, which establishes (c). \square

Example. Consider the endomorphism ϕ of the Heisenberg group H defined by

$$\begin{aligned} \phi(x) &= x^3 y z \\ \phi(y) &= x^{-1} y \end{aligned}$$

Together these imply: $\phi(z) = z^4$, and the endomorphism is represented in this basis by the matrix

$$M = \begin{pmatrix} 3 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 4 \end{pmatrix},$$

with associated weight vector $w = (1, 1, 2)$. The permuted absolute Jordan form of this matrix is

$$M' = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}.$$

The associated divergence rates are then:

$$\begin{aligned} [f_z] &= (3^t)^{\frac{1}{2}} \\ [f_x] &= t \cdot 2^t \\ [f_y] &= 2^t \end{aligned}$$

Thus, the set of divergence rates $\text{div}(\Gamma) = \{t \cdot 3^t, 3^t, 2^t, 3^{\frac{t}{2}}, 2^{\frac{t}{2}}, 2^{\frac{t}{3}}\}$. This is an interesting example, because there are six distinguishable growth rates, as many as there are dimensions of the space.

Proof of the Classification This chapter contains the proofs of three of the main results of this dissertation (Theorems 2-4), which together constitute significant progress towards a classification of nonpolycyclic nilpotent-by-cyclic groups. Section 11 describes a geometric model space for the groups. Sections 12 and 13 contain the proof of Theorem 2. Sections 14-17 contains the proof of Theorem 3. The proof of Theorem 4 is completed in Section 18.

11 Model Spaces for Nilpotent-by-Cyclic Groups

Given a discrete nilpotent group N which is a lattice in a nilpotent Lie group G and an injective endomorphism ϕ of N (which always extends to an endomorphism of G), recall that Γ_ϕ is the HNN extension of N by ϕ , and the Lie group G_ϕ is the semi-direct product $G \rtimes_\phi \mathbb{R} = \{(x, t) \mid x \in G, t \in \mathbb{R}\}$. Multiplication is defined by:

$$(x, t) \cdot (y, s) = (x \cdot_G \phi^t(y), t + s)$$

for all $(x, t), (y, s) \in G \times \mathbb{R}$. Recall that the left-invariant metric on G_ϕ is defined in Section 10.

In this section, we construct a metric complex X_ϕ on which Γ_ϕ acts properly discontinuously and cocompactly by isometries. Thus, Γ_ϕ will be quasi-isometric to the metric space X_ϕ . This construction will parallel the presentation in [FM00].

Let M be the n -manifold with fundamental group $\pi_1(M) = N$. Since Γ_ϕ is an ascending HNN extension of N , it is the fundamental group of the mapping torus of M under the endomorphism ϕ . Let X_ϕ be the universal cover of this mapping torus. Topologically, $X_\phi \approx \mathbb{R}^n \times T_\phi$, where $T = T_\phi$ is the Bass-Serre tree associated to the HNN extension. Thus, T is a homogenous directed tree with one edge oriented in and $[N : \phi(N)]$ edges oriented out of each vertex.

Then X_ϕ is a fiber product of G_ϕ and T_ϕ over \mathbb{R} . The Lie group G_ϕ comes naturally equipped with a height function $h((x, t)) = t$. We define a height function on T_ϕ as follows. Fix a base point x in the tree T_ϕ , and define a path metric on the tree such that each edge has unit length. This gives a height function $h: T_\phi \rightarrow \mathbb{R}$, defined by $|h(y)| = d(x, y)$ and $h(y) > 0$ if and only if the distance minimizing path from x to y begins with an edge which is oriented out of x . The metric on X_ϕ is defined by the fiber product of the metrics on T_ϕ and G_ϕ .

There are induced projections $g_\phi: X_\phi \rightarrow G_\phi$ and $\pi_\phi: X_\phi \rightarrow T_\phi$, and an induced height function $X_\phi \rightarrow \mathbb{R}$.

A *horizontal leaf* $L \subset X_\phi$ is a subset of the form $L = \pi_\phi^{-1}(x)$ where $x \in T_\phi$. Let ℓ be a bi-infinite line in the tree T_ϕ . Then, a *hyperplane* $P_\ell \subset X_\phi$ is a subset of the form $P_\ell = \pi_\phi^{-1}(\ell)$. If the line ℓ is coherently oriented in T_ϕ , then P_ℓ is isometric to G_ϕ by construction, and we call P_ℓ a *coherent hyperplane* in X_ϕ . If the line ℓ is not coherently oriented in T_ϕ (and thus switches orientation exactly once), then we call P_ℓ a *incoherent hyperplane* in X_ϕ . We will show, in Theorem 28, that such a hyperplane is not quasi-isometric to G_ϕ .

Suppose that \mathcal{F} is a decomposition of a metric space X into disjoint subsets whose union is X . Let \mathcal{G} be such a decomposition of a metric space Y . A quasi-isometry $f: X \rightarrow Y$ *coarsely respects* the decompositions \mathcal{F} and \mathcal{G} if there exists an $A \geq 0$ and a map $h: \mathcal{F} \rightarrow \mathcal{G}$ such that for each element $L \in \mathcal{F}$ we have $d_{\mathcal{H}}(f(L), h(L)) \leq A$. For example, we will refer to quasi-isometries which *coarsely respect horizontal leaves* or *coarsely respect vertical flow lines*.

12 One Parameter Jordan Subgroups

In this section and the next, we will prove Theorem 2. Consider the geometric model spaces associated to two nilpotent-by-cyclic groups with the same permuted absolute Jordan form. Since these model spaces are fiber products, it will suffice to show that (1) the associated Lie groups are quasi-isometric, (2) the associated trees (of which the model spaces are fiber products) are quasi-isometric, and (3) the quasi-isometries between them have induced time change functions which are consistent. In this section, we will set the stage for step (1) by establishing a relationship between a matrix and its permuted absolute Jordan form.

Given a matrix $M \in \text{GL}_n(\mathbb{R})$ in Jordan form (not just in real Jordan form—no $J_n(a, b)$ blocks), we say that $\rho(t) = e^{Mt}$ is a 1-parameter *Jordan subgroup*. The matrices e^{Mt} may not be in Jordan form. Not only can a single matrix be conjugated into Jordan form, but Witte has shown ([FM00], Theorem 3.1, also

[Wi]) that an entire 1-parameter subgroup of $\mathrm{GL}_n(\mathbb{R})$ can be transformed into a 1-parameter Jordan subgroup. We will need a corollary of this theorem, under the additional hypothesis that M weakly preserves a grading of \mathbb{R}^n .

Theorem 19 ([FM00], Theorem 3.1, 1-parameter real Jordan form).

Let M^t be a 1-parameter subgroup of $\mathrm{GL}_n(\mathbb{R})$. There exists a 1-parameter Jordan subgroup e^{Jt} , a matrix $A \in \mathrm{GL}_n(\mathbb{R})$ and a bounded 1-parameter subgroup P^t conjugate into the orthogonal group $O(n, \mathbb{R})$, such that e^J is the absolute Jordan form of M , and letting $\overline{M}^t = A^{-1}e^{Jt}A$ we have

$$M^t = \overline{M}^t P^t = P^t \overline{M}^t.$$

Proof. (from [FM00], p. 156) Given a general 1-parameter subgroup $e^{\mu t}$ in $\mathrm{GL}_n(\mathbb{R})$, choose A so that $A^{-1}\mu A$ is in real Jordan form, and so $A^{-1}\mu A = \delta + \nu + \eta$ where δ is diagonal, ν is superdiagonal, and η is skew-symmetric. Let $B = e^A$, so that

$$e^{\mu t} = (Be^{(\delta+\nu)t}B^{-1})(Be^{\eta t}B^{-1}).$$

Since η is skew symmetric it follows that $e^{\eta t}$ is in the orthogonal group $O_n(\mathbb{R})$. \square

It is a surprising but useful fact that $Ae^B A^{-1} = e^{ABA^{-1}}$.

The theorem concludes that P^t is a bounded subgroup. That is,

$$\sup_{x \in \mathbb{R}^n} \frac{|P^t x|}{|x|}$$

is bounded for all t , where $|\cdot|$ is the usual Euclidean norm. This does not imply that, using the nilpotent norm $\|\cdot\|$, $\sup_{x \in \mathbb{R}^n} \frac{\|P^t x\|}{\|x\|}$ is bounded for all t , or even that the sup is finite for a fixed t . For example, consider the Heisenberg group in $\{x, y, z\}$ basis, with $P \cdot z = x$. Then, $\frac{\|P(kz)\|}{\|kz\|} = \frac{k}{\sqrt{k}}$. It is precisely this type of map, which fails to preserve the Carnot grading, for which $\frac{\|Px\|}{\|x\|}$ fails to be bounded.

In Section 13, we will use Corollary 20 to show that, with the additional condition that M weakly preserves the nilpotent grading, this type of mixing is excluded.

Corollary 20. *Under the conditions of Theorem 19, assume further that matrix M weakly preserves the grading $\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$, with $\dim(V_i) = n_i$. Then P^t is conjugate into the product of the corresponding orthogonal groups $O_{n_1}(\mathbb{R}) \times \cdots \times O_{n_k}(\mathbb{R})$.*

Proof. Let $A \in \mathrm{GL}_n(\mathbb{R})$ be the matrix such that $A^{-1}MA$ is in Jordan form. Suppose that $V \subset \mathbb{R}^n$ is preserved by M . Denote $d = \dim(V)$ and choose a basis $\{e_1, \dots, e_n\}$ for \mathbb{R}^n such that

$$V = \mathrm{span}\{e_1, \dots, e_d\}.$$

Express matrix M with respect to this basis. We define a truncated matrix $M' \in \mathcal{M}_d(\mathbb{R})$ as follows:

$$(M')_{i,j} = (M)_{i,j} \text{ if } 1 \leq i, j \leq d.$$

Let $i : V \rightarrow \mathbb{R}^n$ be the inclusion map. Then, for any $v \in V$, $i(M'v) = Mv$. Applying Theorem 19 to matrix M' yields:

$$(M')^t = (\overline{M}')^t (P')^t = (P')^t (\overline{M}')^t,$$

where $\overline{M}', P' \in \mathcal{M}_d(\mathbb{R})$. In particular, both \overline{M}' and P' preserve V . Since the action of M and M' are identical on V , the action of P and P' must also agree on V . Thus, P must preserve V . Because P is orthogonal, it must also preserve V^\perp . Thus, it acts orthogonally on each component, and $P \in O_d(\mathbb{R}) \oplus O_{n-d}(\mathbb{R})$. Applying this argument to each $V_i \oplus \cdots \oplus V_k$ shows that $P \in O_{n_1}(\mathbb{R}) \times \cdots \times O_{n_k}(\mathbb{R})$. \square

The Jordan form, and thus the absolute Jordan form, of a matrix is unique up to permutation of the blocks. When the matrix M represents a linear transformation of \mathbb{R}^n , conjugating represents determining a new choice of basis for the space \mathbb{R}^n , and permuting the blocks corresponds to permuting the elements of the basis.

13 Reducing a Matrix to Permuted Absolute Jordan Form

In this section we will complete the proof of Theorem 2 by first showing that the Lie groups are quasi-isometric. This step is mostly handled by the following theorem, which corresponds to Proposition 4.1 in [FM00] and is restated here in the setting of nilpotent-by-cyclic groups. The proof requires more subtlety in this situation.

Theorem 21 (Quasi-isometric Lie Groups). *Let ϕ and θ be injective endomorphisms of a fixed nilpotent Lie group G . Suppose that the maps of the Lie algebra \mathfrak{g} induced by ϕ and θ are represented by matrices M and N which lie on 1-parameter subgroups M^t, N^t in $\mathrm{GL}(n, \mathbb{R})$. Suppose there exist integers $r, s > 0$ such that M^r and N^s have the same permuted absolute Jordan form. Then the metric spaces G_ϕ and G_θ are quasi-isometric. To be explicit, there exists $A \in \mathrm{GL}(n, \mathbb{R})$ and $K \geq 1$ such that for each $t \in \mathbb{R}$, the map $v \mapsto A(v)$ is a K -bilipschitz homeomorphism from the metric $d_{\phi,t}$ to the metric $d_{\theta, \frac{s}{r} \cdot t}$; it follows that the map from $G_\phi = \mathbb{R}^n \rtimes_\phi \mathbb{R}$ to $G_\theta = \mathbb{R}^n \rtimes_\theta \mathbb{R}$ given by*

$$(x, t) \mapsto \left(Ax, \frac{s}{r} \cdot t \right)$$

is a bilipschitz homeomorphism from G_ϕ to G_θ , with bilipschitz constant $\sup\{K, \frac{s}{r}, \frac{r}{s}\}$.

The proof of this theorem in the setting of nilpotent-by-cyclic groups will require the following two lemmas, which provide information about the effect of A and P^t on the nilpotent geometry.

Lemma 22 (ϕ is Bounded.). *Suppose $\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$ is the Lie algebra of a Carnot group, and that $\phi: \mathfrak{g} \rightarrow \mathfrak{g}$ is an injective endomorphism. Then $\frac{||\phi(v)||}{||v||}$ is bounded away from 0 and ∞ .*

Proof. Define $\phi_{i,j}: V_i \rightarrow V_j$ as follows:

$$\phi_{i,j}(v_i) = \text{proj}_{V_j}(\phi(\text{inc}_i(v_i))),$$

where inc_i is the inclusion of V_i in V . For each $\phi_{i,j}$, define:

$$m_{i,j} = \min_{v_i \in V_i \setminus \{0\}} \left\{ \frac{|\phi_{i,j}(v_i)|}{|v_i|} \right\}, \text{ and } M_{i,j} = \max_{v_i \in V_i \setminus \{0\}} \left\{ \frac{|\phi_{i,j}(v_i)|}{|v_i|} \right\}.$$

The minimum and maximum always exist, although they may be zero. Lemma 9 states that ϕ weakly preserves the grading of \mathfrak{g} by $\{V_i\}$. As a result, $\phi_{i,j} = 0$ for $i > j$. Because ϕ is also injective and \mathfrak{g} is finite dimensional, $m_{i,i} \neq 0$ for all $i = 1, \dots, k$. **Bounded Above:** Now consider $x = (x_1, \dots, x_k)$, where each $x_i \in V_i$. Then,

$$\begin{aligned} \left| \text{proj}_{V_j} \phi(x) \right| &= |\Sigma_{1 \leq i \leq j} \phi_{i,j}(x_i)| \\ &\leq k \cdot \max_{1 \leq i \leq j} |\phi_{i,j}(x_i)| \\ &\leq k \cdot \max_{1 \leq i \leq j} M_{i,j} |x_i|. \end{aligned}$$

Corollary 7 implies there is $K > 0$ such that:

$$\begin{aligned} ||\phi(x)|| &\leq K \max_j \left\{ \sqrt[j]{(\phi(x))_j} \right\} \\ &\leq K \max_j \left\{ k \max_{1 \leq i \leq j} \sqrt[j]{M_{i,j} |x_i|} \right\} \\ &\leq K \cdot k \cdot \max_{i,j} \sqrt[j]{(M_{i,j})} \cdot \max_j \max_{1 \leq i \leq j} \sqrt[j]{|x_i|}. \end{aligned}$$

Let $M = \max_{i,j} \left\{ \sqrt[j]{M_{i,j}} \right\}$. Observe also that, for fixed i ,

$$\max_{j \geq i} \sqrt[j]{|x_i|} = \sqrt[i]{|x_i|}.$$

Thus,

$$\begin{aligned} ||\phi(x)|| &\leq K \cdot k \cdot M \cdot \max_i \left\{ \sqrt[i]{|x_i|} \right\} \\ &\leq K \cdot k \cdot M \cdot ||x||, \end{aligned} \tag{3}$$

so $\frac{\|\phi(x)\|}{\|x\|}$ is bounded above.

Bounded Below: By definition of $m_{i,i}$,

$$|\text{proj}_{V_i} \phi(x)| \geq m_{i,i} |x_i|.$$

Define

$$m = \min_i \left\{ \sqrt[i]{m_{i,i}} \right\},$$

and note that $m > 0$. Again, considering the nilpotent metric:

$$\begin{aligned} \|\phi(x)\| &> \frac{1}{K} \max_i \left\{ \sqrt[i]{\text{proj}_{V_i} \phi(x)} \right\} \\ &\geq \frac{1}{K} \max_i \left\{ \sqrt[i]{m_{i,i} |x_i|} \right\} \\ &\geq \frac{1}{K} \cdot m \cdot \max_i \left\{ \sqrt[i]{|x_i|} \right\} \\ &\geq \frac{1}{K} \cdot m \cdot \frac{\|x\|}{K}. \end{aligned} \tag{4}$$

Thus, $\frac{\|\phi(x)\|}{\|x\|}$ is bounded away from both 0 and ∞ . \square

Lemma 23 (P is Bounded on \mathfrak{g}). *Suppose $\mathfrak{g} = V_1 \oplus \cdots \oplus V_k$ is the Lie algebra of a Carnot group. Suppose further that $P \in O(n_1) \oplus \cdots \oplus O(n_k)$, where $n_i = \dim(V_i)$ and that P^t is bounded on \mathbb{R}^n . Then*

$$\sup_{x \in G} \frac{\|P^t x\|}{\|x\|}$$

is uniformly bounded; i.e., the bound is independent of t .

Proof. Lemma 22 shows that, for fixed t , $\sup_{x \in \mathbb{R}^n} \frac{\|P^t x\|}{\|x\|}$ is bounded. In fact, Equations 3 and 4 show:

$$\frac{m}{K^2} \leq \frac{\|\phi(x)\|}{x} \leq MKk,$$

where K and k depend only on the group G . Although M and m depend on P^t , the fact that P^t is bounded in the Euclidean norm implies that M and m are uniformly bounded for all t . This implies the lemma. \square

Proof of Theorem 21. The proof proceeds similarly to the proof of Proposition 4.1 in [FM00] but the nonisotropic nature of the nilpotent geometry makes the proof more involved.

Case 1: Assume that $N^t = e^{Jt}$ is the unique 1-parameter subgroup such that $N = e^J$ is conjugate to the absolute Jordan form of M . Then by Theorem 19 and its Corollary 20

$$M^t = (A^{-1} N^t A) P^t$$

where $A \in \mathrm{GL}_n(\mathbb{R})$ and the 1-parameter subgroup P^t is a bounded element of the product of the orthogonal groups $O(n_1) \oplus \cdots \oplus O(n_k)$. Choose $t \in \mathbb{R}$ and $v \in \mathbb{R}^n$. We must show that the two numbers

$$\|M^{-t}v\| = \|P^{-1}(A^{-1}N^tA)v\| \text{ and } \|N^{-t}Av\|$$

have ratio bounded away from 0 and ∞ , with bound independent of t, v . We set $u = N^{-t}Av$, so it suffices to show that $\|P^{-t}A^{-1}u\|$ and $\|u\|$ have bounded ratio.

As discussed in Section 12, this shall be more difficult than in the abelian-by-cyclic case. However, Lemma 23 shows that if $P \in O(n_1) \oplus \cdots \oplus O(n_k)$ and P^t is bounded, then $\frac{\|P^t v\|}{\|v\|}$ remains bounded. Furthermore, as shown in Lemma 22, the ratio $\frac{\|Ax\|}{\|x\|}$ is also bounded away from both 0 and ∞ for the matrix A .

Case 2: Assume that there exists $a > 0$ such that $M^t = N^{at}$ for all t . Then the metrics $d_{M,t}$ and $d_{N,at}$ are identical.

General case: Applying Case 2 we may assume that $\det M = \det N$. Applying Case 1 twice we may go from G_M to G_{e^J} to G_N , where e^J is conjugate to the absolute Jordan form of M and of N . \square

We now have the tools to complete the proof of Theorem 2.

Proof of Theorem 2. Given an endomorphism ϕ of a nilpotent group N , the nilpotent-by-cyclic group Γ_{ϕ^k} is a finite index subgroup of Γ_ϕ , and thus they are quasi-isometric. Now we can restrict our attention to groups defined by endomorphisms with the same permuted absolute Jordan form.

Suppose N_1 and N_2 are lattices in the same Carnot group G and ϕ_1 and ϕ_2 are injective, nonsurjective endomorphisms of N_1 and N_2 respectively, each acting without unipotent part. Suppose that M_1 and M_2 have the same permuted absolute Jordan form. By Theorem 21, the Lie groups associated to M_1 and M_2 are quasi-isometric via a height preserving quasi-isometry. It remains to show there is a height preserving quasi-isometry between the associated trees.

If M' is the permuted absolute Jordan form of M , then

$$\det(M') = |\det(M)|.$$

Therefore, if M_1 and M_2 have the same permuted absolute Jordan form, then their determinants have the same absolute value.

As described in Section 11, the tree associated to Γ_i is uniform with 1 ‘in’ and $[N : \phi(N)]$ ‘out’ edges at each vertex. The index $[N : \phi(N)] = |\det(M)|$, where M is the matrix which describes the induced action of ϕ on the Lie algebra. Thus, there is a height preserving isometry between the Bass-Serre trees associated to Γ_1 and Γ_2 .

Recall that the group Γ_i is quasi-isometric to the fiber product X_i of the Lie group G_i and the tree T_i . The height preserving quasi-isometries of the Lie group and the tree induce a quasi-isometry of the fiber product which shows that the nilpotent-by-cyclic groups Γ_{ϕ_1} and Γ_{ϕ_2} are quasi-isometric. \square

14 Coarse Topology, inducing a Quasi-isometry of G_ϕ

In this section, we will establish the following:

Proposition 24. *Suppose that N is a discrete nilpotent group which is a lattice in a nilpotent Lie group G . Let ϕ be an injective nonsurjective endomorphism of N . Recall that $\Gamma_{N,\phi}$ is the discrete group which is the HNN extension of N by ϕ . Similarly, consider N' , G' , ϕ' , and $\Gamma_{N',\phi'}$ similarly. If there exists a quasi-isometry $f: \Gamma_{N,\phi} \rightarrow \Gamma_{N',\phi'}$ then:*

- a) *There is a quasi-isometry between the nilpotent groups $g: N \rightarrow N'$.*
- b) *There is a quasi-isometry between the Lie groups $\theta: G_\phi \rightarrow G'_{\phi'}$ which coarsely respects the transversely oriented horizontal foliations.*
- c) *Furthermore, all associated constants for θ depend only on those for f .*

This is a modified version of [FM00], Proposition 7.1. The proof there proceeds in four steps. Our presentation will parallel the one there, with Steps 1 and 2 (as encapsulated in Theorems 7.3 and 7.7, respectively) as well as Step 4 applying directly. Step 3 will require a new proof, given below in Theorem 28. The equivalent of part (a) in the abelian-by-cyclic case required only showing that dimension was preserved. Here, this part is a new Corollary of their Theorem 7.7.

Proof. Consider a quasi-isometry between two nilpotent-by-cyclic groups $f: \Gamma_{N,\phi} \rightarrow \Gamma_{N',\phi'}$. This induces a quasi-isometry between the geometric model spaces X_ϕ and $X_{\phi'}$. Henceforth, we will think of these spaces and maps interchangeably.

Step 1. Quasi-isometrically embedded hyperplanes are close to hyperplanes. In the language of [FM00], the HNN extensions $\Gamma_{N,\phi}$ are finite, geometrically homogenous graphs of groups. Furthermore, the edge and vertex groups are fundamental groups of Poincare duality spaces. This is because (the universal cover) G is connected and simply-connected, and therefore homeomorphic to \mathbb{R}^n . Also, $\pi_1(G/\Gamma) = \Gamma$. Since the manifolds G/Γ satisfy Poincare duality, each Γ is a Poincare duality group. Thus, as discussed in [FM00], the following theorem applies to the metric fibration of the geometric model space over the associated tree: $X_{N,\phi} \rightarrow T_{N,\phi}$.

Theorem 25 ([FM00], Theorem 7.3). *Let $\pi: X \rightarrow T$ be a metric fibration whose fibers are contractible n -manifolds for some n . Let P be a contractible $(n+1)$ -manifold which is a uniformly contractible, bounded geometry, metric simplicial complex. Then for any uniformly proper embedding $\phi: P \rightarrow X$, there exists a unique hyperplane $Q \subset X$ such that $\phi(P)$ and Q have finite Hausdorff distance in X . The bound on Hausdorff distance depends only on the metric fibration data for π , the uniform contractibility data and bounded geometry data for P , and the uniform properness data for ϕ .*

So, each hyperplane of $X_{N,\phi}$ is mapped by f to a (universally) bounded neighborhood of some hyperplane $P_{\ell'} \subset X_{N',\phi}$.

Step 2. A quasi-isometry takes hyperplanes and horizontal leaves in X_ϕ to hyperplanes and horizontal leaves in $X_{\phi'}$: This is the step in the proof which depends upon having endomorphisms of the discrete groups N, N' which are *not* surjective. This implies that the tree $T = T_{N,\phi}$ (respectively, $T' = T_{N',\phi'}$) has uniform valence $[N : \phi(N)] > 1$ (respectively, $[N' : \phi'(N')] > 1$). For $\beta > 0$ we say a tree T is β -bushy if each point of T is within distance β of some vertex v such that $T - \{v\}$ has at least 3 unbounded components. So, the trees T and T' are $\frac{1}{2}$ -bushy, in the path metric described in Section 11.

Thus, the following theorem applies:

Theorem 26 ([FM00], Theorem 7.7). *Let $\pi: X \rightarrow T, \pi': X' \rightarrow T'$ be metric fibrations over β -bushy trees T, T' , such that the fibers of π and π' are contractible n -manifolds for some n . Let $f: X \rightarrow X'$ be a quasi-isometry. Then there exists a constant A , depending only on the metric fibration data of π, π' , the quasi-isometry data for f , and the constant β , such that:*

- a) *For each hyperplane $P \subset X$ there exists a unique hyperplane $Q \subset X'$ such that $d_{\mathcal{H}}(f(P), Q) \leq A$.*
- b) *For each horizontal leaf $L \subset X$ there is a horizontal leaf $L' \subset X'$ such that $d_{\mathcal{H}}(f(L), L') \leq A$.*

As an immediate consequence, we have:

Corollary 27 (Quasi-isometric Base Groups). *If $\Gamma_{N,\phi}$ and $\Gamma_{N',\phi'}$ are nilpotent-by-cyclic groups which are quasi-isometric, then N and N' are quasi-isometric, and there is a quasi-isometry $f: G_{N,\phi} \rightarrow G_{N',\phi'}$ which coarsely respects the horizontal foliations and their transverse orientations.*

Proof. Let $X = X_{N,\phi}$ be the geometric model space for $\Gamma_{N,\phi}$, and X' be the model space for $\Gamma_{N',\phi'}$. If $f: X \rightarrow X'$ is a (K, C) -quasi-isometry, then Theorem 26 implies that there is $A > 0$ such that for every horizontal leaf $L \subset X$, there is a horizontal leaf $L' \subset X'$ such that $d_{\mathcal{H}}(f(L), L') \leq A$. Let $p: X' \rightarrow L'$ be nearest point projection. Then, $p \circ f: L \rightarrow L'$ is a $(K, C + 2A)$ -quasi-isometry. Furthermore, $N_{2A}(p \circ f(L)) \supset L'$. Thus $p \circ f$ shows that L and L' are quasi-isometric. Since L is quasi-isometric to the nilpotent discrete group N , and L' is quasi-isometric to the nilpotent discrete group N' , we conclude that N and N' are quasi-isometric. \square

This result shows that, in classifying nilpotent-by-cyclic groups, we may restrict our attention to groups which are extensions of nilpotent groups which are quasi-isometric. As mentioned in the introduction, the quasi-isometric classification of nilpotent groups is a major unsolved problem. For the classification given here, we will make the slightly stronger

Standing Assumption: Discrete nilpotent groups N and N' are lattices in the same Carnot group G .

Step 3. A quasi-isometry takes coherent hyperplanes in X_ϕ to coherent hyperplanes in $X_{\phi'}$.

Theorem 28 (Coherence is a quasi-isometry invariant). *Given a quasi-isometry between hyperplanes $f : H_1 \rightarrow H_2$, the hyperplane H_1 is coherent if and only if H_2 is coherent.*

As in the abelian-by-cyclic case, the proof of this step is based on a comparison of the growth functions for coherent and incoherent hyperplanes. For coherent hyperplanes, the growth will be linear (it is quadratic in the abelian-by-cyclic case), while for incoherent hyperplanes it will be exponential.

Proof. We need the following definitions as in [FM00]: For any hyperplane $H \subset X$, there is a quotient map $H \rightarrow \mathbb{R}$ whose point pre-images give the horizontal foliation of H , and such that the Hausdorff distance between two horizontal leaves equals the distance between the corresponding points in \mathbb{R} . A path γ in H is said to be (K, C) -*quasivertical* if its projection to \mathbb{R} is a (K, C) -quasigeodesic. Define a (K, C) -*quasivertical bigon* in H to be a pair of (K, C) -quasivertical paths γ, γ' which begin and end at the same point.

If K, C are fixed, we define a filling area function $A(L)$ for (K, C) -quasivertical bigons in H . Given a (K, C) -quasivertical bigon γ, γ' , its *filling area* is the infimal area of a Lipschitz map $D^2 \rightarrow H$ whose boundary is a reparameterization of the closed curve $\gamma^{-1} * \gamma'$; such a map $D^2 \rightarrow H$ is called a *filling disc* for $\gamma^{-1} * \gamma'$. For each $L \geq 0$ define $\mathcal{A}(L)$ to be the supremal filling area over all (K, C) -quasivertical bigons γ, γ' in H such that $\text{Length}(\gamma) + \text{Length}(\gamma') \leq L$.

Suppose that two hyperplanes H_1 and H_2 have filling functions $\mathcal{A}_1(L)$ and $\mathcal{A}_2(L)$, respectively. As is shown in [FM00], if there is a quasi-isometry between H_1 and H_2 , then the filling functions must be comparable in the following sense:

$$\mathcal{A}_1(L) \leq \alpha \cdot \mathcal{A}_2(\beta L + \delta) + \zeta, \quad (5)$$

for some $\alpha, \beta, \delta, \zeta > 0$ which do not depend on L , and the same equation must hold with \mathcal{A}_1 and \mathcal{A}_2 reversed.

However, we will see that in the geometric model space associated to a nilpotent-by-cyclic group, the filling function of a coherent hyperplane is linear, while the filling function of an incoherent hyperplane is exponential.

Consider a (K, C) -quasivertical bigon in a coherent hyperplane. As shown in Section 16, a quasivertical line must be shadowed by a vertical flow line. (This is a result of the assumption that the endomorphism is unipotent-free; without this assumption, we could get quadratic growth, as in the abelian-by-cyclic case.) Thus, the distance between the two edges of the bigon is universally bounded. Therefore, the filling area is linear in L .

In the case of an incoherent hyperplane, the argument used in [FM00] depends only on having one direction in which the growth of vectors is exponential with base greater than 1. This is true in the nilpotent-by-cyclic situation, although the growth is calculated as $\lambda^{\frac{1}{w_i}}$ and not simply λ .

Since linear and exponential functions are not comparable according to equation 5, there must not be a quasi-isometry between coherent and incoherent hyperplanes. \square

Step 4. A horizontal-respecting quasi-isometry preserves transverse orientation. The proof in [FM00] applies directly. The idea of the proof is this: a quasi-isometry must preserve the sign of the log of the determinant.

That is, given a quasi-isometry $\phi: \Gamma_1 \rightarrow \Gamma_2$, the absolute values of the determinants of the matrices associated to Γ_1 and Γ_2 must either both be greater than one or both less than one. Since reversing orientation comes from taking the inverse of the matrix, this shows that the quasi-isometry must preserve orientation.

This concludes the proof of Proposition 24. \square

15 Time Change Rigidity, Part 1

Recall that a map f between hyperplanes P and P' is *horizontal-respecting* if there exists a function $h: \mathbb{R} \rightarrow \mathbb{R}$ and $A \geq 0$ such that $d_{\mathcal{H}}(f(P_t), P'_{h(t)}) \leq A$ for all $t \in \mathbb{R}$, where $P_t = \pi^{-1}(\ell \cap \{t\})$. The function $h: \mathbb{R} \rightarrow \mathbb{R}$ is called an *induced time change* for f with *Hausdorff constant* A . If h and h' are two induced time changes for f with Hausdorff constants A and A' , then $\sup_t |h(t) - h'(t)| \leq A + A'$. The converse is also true: If h is an induced time change for f with Hausdorff constant A , and $h': \mathbb{R} \rightarrow \mathbb{R}$ satisfies $\sup_t |h(t) - h'(t)| \leq A'$, then h' is also an induced time change function for f with Hausdorff constant $A + A'$.

Let f be the quasi-isometry between hyperplanes which is guaranteed by Corollary 24. Later (Section 17), we will show that there is a linear function which is an induced time change function for f . That proof will depend upon the result of Section 16: that vertical flow lines are coarsely preserved. To show that, however, we need the fact that the induced time change is at least coarsely linear and coarsely increasing; that is, $h: \mathbb{R} \rightarrow \mathbb{R}$ is a quasi-isometry and $h(t) \rightarrow \infty$ as $t \rightarrow \infty$. The proof of the following Lemma, given by Farb and Mosher ([FM00], Lemma 5.1) for the abelian-by-cyclic case, applies in this context as well.

Lemma 29. *For each K, C, A there exists C' such that if $f: G_\phi \rightarrow G_{\phi'}$ is a horizontal respecting (K, C) quasi-isometry, and $h: \mathbb{R} \rightarrow \mathbb{R}$ is an induced time change for f with Hausdorff constant A , then h is a (K, C') quasi-isometry of \mathbb{R} .*

16 Vertical Flow Lines are Coarsely Preserved

In this section, we will show that the quasi-isometry f between $G_{N, \phi}$ and $G_{N', \phi'}$ established by Corollary 24 coarsely preserves vertical flow lines. As explained in [FM00], if all the eigenvalues of M and N , the matrices associated to ϕ and ϕ' , respectively, are greater than 1, then G_M and G_N are negatively curved,

and this is simply the fact that a quasigeodesic in a negatively curved space is Hasudorff close to a geodesic. Since we permit M to have any eigenvalue with absolute value different from 1, we need the stronger:

Theorem 30. *Consider a quasi-isometry $f: G_\phi \rightarrow G_{\phi'}$. There exists $\alpha \geq 0$ such that for each vertical flow line l_x in G_ϕ , there exists a vertical flow line m_y in $G_{\phi'}$ such that $f(l_x)$ is contained in the α -neighborhood of m_y .*

The proof given in [FM00] (Claim 5.7, p.165-6) goes through exactly as given. The idea of the proof is given below, following some definitions. In that context, the theorem is stated for leaves of the center foliation, but with our additional assumption that Γ has no center (implied by the standing assumption that ϕ is unipotent-free), the center leaves are simply vertical flow lines.

Definition. Given a flow Φ on a metric space X , we write $x \cdot t$ as an abbreviation for $\Phi_t(x)$. Given $\epsilon, T \geq 0$, an (ϵ, T) -pseudo-orbit of Φ consists of a sequence of flow segments $(x_i \cdot [0, t_i])$, where the index i runs over an interval in \mathbb{Z} , such that $d_X(x_i \cdot t_i, x_{i+1}) \leq \epsilon$ and $t_i \geq T$ for all i .

A flow Φ on a manifold M is called *hyperbolic* if, at each $x \in M$, the tangent space has a splitting $T_x M = E^u \oplus E^s$ such that the derivative $d\Phi: TM \rightarrow TM$ expands E^u , contracts E^s , and preserves both. The subbundles E^s and E^u are tangent to the global stable and unstable manifolds, defined by:

$$W^s(x) = \{y \in M \mid d(f^n(x), f^n(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}, \text{ and}$$

$$W^u(x) = \{y \in M \mid d(f^{-n}(x), f^{-n}(y)) \rightarrow 0 \text{ as } n \rightarrow \infty\}.$$

Given these structures, it can be shown (e.g., [KH] Proposition 6.4.13) that:

- there is an $\epsilon > 0$ such that for any $x, y \in X$, the intersection $W^s(x) \cap N_\epsilon(x) \cap W^u(y) \cap N_\epsilon(y)$ consists of at most one point, and
- there is a $\delta > 0$ such that $W^s(x) \cap N_\epsilon(x) \cap W^u(y) \cap N_\epsilon(y) \neq \emptyset$ whenever $d(x, y) < \delta$.

Such a structure is called a *local product structure*. An equivalent statement is: there is a $\delta > 0$ and, for all $p \in X$, a map $f_p: N_\delta(p) \rightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$ such that $f(W^u(x)) \subset \mathbb{R}^k \times \{v\}$ for some $v \in \mathbb{R}^{n-k}$ and $f(W^s(x)) \subset \{w\} \times \mathbb{R}^{n-k}$ for some $w \in \mathbb{R}^k$.

If there is a single map $f: M \rightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$ which satisfies the above conditions, for $\epsilon, \delta = \infty$ then we say f has a *global product structure*.

The existence of a local product structure for hyperbolic flows is a key ingredient in the proof of the so-called Shadowing Lemma. This was first proved by Bowen ([Bow], Theorem 2.2, Approximation Theorem). Hirsch, Pugh, and Shub found a new proof for the following more streamlined statement:

Lemma 31 (Shadowing Lemma, [HPS] Lemma 7A.2, p. 133). *If (f, L) has local product structure and $\nu > 0$ is given, then there exists $\delta > 0$ such that any δ -pseudo orbit for f in Λ can be ν -shadowed by a pseudo-orbit for f in which Λ which respects L .*

Notice the quantifiers: in this statement (as in Bowen's presentation), δ depends on ν , and as ν approaches zero, so does δ .

In Bowen's proof, dependence of δ on ν (ϵ in his statement) arises in two different ways. First, he needs local product structures on neighborhoods with radius at least δ . This dependence is not surprising; it says that we must have a product structure on neighborhoods which are large enough that the pseudo-orbit cannot jump out of the neighborhood. Secondly, he requires that δ be small enough to satisfy a bound on the sum of a particular convergent geometric sequence. This sequence also depends upon $\min\{\lambda > 1, \lambda^{-1} > 1 \mid \lambda \text{ is an eigenvalue of } df\}$. This condition can be satisfied for a fixed δ by raising M to large enough powers.

In applying this lemma here (as in [FM00]), we want to choose an arbitrarily large δ . This can be accomplished if the requirement of a local product structure is replaced by the condition of having a global product structure. Farb and Mosher saw that the existence of a global product structure is sufficient to ensure that, for arbitrary choice of δ , there is a ν so that the conclusion of the Shadowing Lemma holds. They stated and used the following "Global Shadowing Lemma". (It is stated here in the special case that center leaves are vertical flow lines, which is true for nilpotent-by-cyclic groups.)

Lemma 32 (Shadowing Lemma, [FM00] Lemma 5.3, p. 163). *Consider a 1-parameter subgroup M^t of $GL_n(\mathbb{R})$, and let Φ be the vertical flow on G_M . For every $\epsilon, T \geq 0$, there exists $\delta \geq 0$ such that every (ϵ, T) -pseudo-orbit of Φ is δ -shadowed by a vertical flow line m_y . That is, if $(x_i \cdot [0, t_i])$ is an (ϵ, T) -pseudo-orbit, then there is a vertical line m_y such that $d(x_i \cdot t, y \cdot t) \leq \delta$ for all i and all $t \in [0, t_i]$.*

In the cases of abelian-by-cyclic and nilpotent-by-cyclic groups, a global product structure exists for any hyperbolic flow on G . On the Lie algebra \mathfrak{g} of G , the global stable manifold W^s is just the span of the root spaces with eigenvalues less than 1; the global unstable manifold is the span of the eigenspaces greater than 1. Here, the global product structure is clear; the leaves are all linear subspaces, and $\dim(W^s) + \dim(W^u) = n$, so any stable manifold intersects any unstable manifold at a unique point.

The exp map is a diffeomorphism from \mathfrak{g} to G which conjugates df to f . Thus, it carries stable manifolds of \mathfrak{g} to stable manifolds of G and preserves the global product structure. Thus, G has a global product structure as well.

Under our condition of unipotent-free endomorphisms, this gives a global decomposition of the space. Thus, Farb and Mosher's Global Shadowing Lemma applies in the situation of nilpotent-by-cyclic groups.

Idea of Proof of Theorem 30. The proof is in two steps. First, show that $f(l_x)$ is close to a pseudo-orbit. To do this, use the fact that there is an coarsely-linear induced time change which is coarsely increasing (Lemma 29) to choose a sparse but regularly spaced sequence $\{x_i\}$ in the flow l_x . The images $\{y_i = f(x_i)\}$ of these points are used to define a pseudo-orbit in $G_{\phi'}$. Then, the Shadowing

Lemma implies that the pseudo-orbit is within a bounded neighborhood of a vertical flow line. \square

17 Time Change Rigidity, Part 2

In Section 15, we showed that the induced time change was coarsely linear and coarsely increasing. Now, with the additional information that vertical flow lines are coarsely preserved, we can show more:

Theorem 33 (Linear Induced Time Change). *Suppose f is a horizontal-respecting quasi-isometry between Lie groups G_ϕ and $G_{\phi'}$, both of which satisfy our standing hypotheses. (That is, ϕ and ϕ' are unipotent-free nonsurjective endomorphisms of discrete nilpotent groups N which are lattices in the same Lie group G .) Then, there exists $m > 0$ such that $h(t) = mt$ is an induced time change for f .*

The proof relies upon comparing the divergence rates of vertical flow lines in each Lie group. The corresponding result in the abelian-by-cyclic case ([FM00], Prop. 5.8) is similar but can be proved much more simply. In that case, the smallest divergence rate which is greater than the constant must be a pure exponential. That is, it must have the form λ^t . In that case, if α is the eigenvalue of G_ϕ with minimal absolute value greater than 1, and β is the corresponding eigenvalue for $G_{\phi'}$, then the slope of the induced time change is $m = \frac{\log \alpha}{\log \beta}$.

Much more work will be required in the nilpotent-by-cyclic case. It seems true, but would take significant work to show, that the smallest growth rate is pure exponential. (Perhaps the smallest growth rate is $(t^k \lambda^t)^{\frac{1}{w}}$ for some $k \neq 0$, while the vectors which grow as λ^t are in a grade of the nilpotent group with smaller weight w .) Instead, we compute the induced time change which would be implied if a line with pure exponential growth were taken instead to a polynomial-exponential diverging line. Considering the coarse inverse of f , we find a similar rate for the inverse. Putting these together shows that h must be linear.

The following Lemma establishes induced time change parameters, given the divergence rates of flow lines which are preserved by the quasi-isometry.

Lemma 34. *Consider a horizontal respecting, vertical flow line preserving (K, C) -quasi-isometry $f : G_\phi \rightarrow G_{\phi'}$ for $K \geq 1, C \geq 0$. Let g be the map between the vertical line spaces, (with constant R) and suppose that $g(\ell_1) = m_1$ and $g(\ell_2) = m_2$, where ℓ_i is a vertical flow line in G_ϕ and m_i is a vertical flow line in $G_{\phi'}$. Suppose further that $d(\ell_1(t), \ell_2(t)) \simeq t^k \cdot \mu^t$ and $d(m_1(t), m_2(t)) \simeq \lambda^t$ with $k > 0$ and $\lambda, \mu > 1$. Let $K_\ell \geq 1$ and $C_\ell \geq 0$ be the constants implicit in the equation $d(\ell_1(t), \ell_2(t)) \simeq t^{K_\ell} \cdot \mu^{C_\ell t}$, and define K_m, C_m similarly. Then, there are positive constants m, c such that the function $h(t) = mt + c \log(t)$ is an induced time change for f .*

Proof. We will find two different bounds for the distance $d(f(\ell_1(t)), f(\ell_2(t)))$, which, when compared, give restrictions on the induced time change function.

Suppose that $h(t)$ is an induced time change function for f with Hausdorff constant A . (One must exist, since f is horizontal foliation preserving.) Then,

$$d(f(\ell_i(t)), m_i(h(t))) < R + A.$$

Consequently,

$$|d(m_1(h(t)), m_2(h(t))) - d(f(\ell_1(t)), f(\ell_2(t)))| \leq 2R + 2A,$$

and applying the known divergence rate of m_i yields:

$$\frac{1}{K_m} \mu^{h(t)} - C_m - 2R - 2A \leq d(f(\ell_1(t)), f(\ell_2(t))) \leq K_m \mu^{h(t)} + C_m + 2R + 2A. \quad (6)$$

On the other hand, by definition of quasi-isometry, we have

$$\frac{1}{K} \cdot d(\ell_1(t), \ell_2(t)) - C \leq d(f(\ell_1(t)), f(\ell_2(t))) \leq K \cdot d(\ell_1(t), \ell_2(t)) + C,$$

and substituting the divergence rate of ℓ_i yields:

$$\frac{1}{KK_\ell} \cdot t^k \lambda^t - \frac{C_\ell}{K} - C \leq d(f(\ell_1(t)), f(\ell_2(t))) \leq KK_\ell \cdot t^k \lambda^t + KC_\ell + C. \quad (7)$$

Thus, the upper bound of Equation 6 must be greater than the lower bound of Equation 7, and vice versa. That is,

$$K_m \mu^{h(t)} + C_m + 2R + 2A \geq \frac{1}{KK_\ell} \cdot t^k \lambda^t - \frac{C_\ell}{K} - C$$

and

$$KK_\ell \cdot t^k \lambda^t + KC_\ell + C \geq \frac{1}{K_m} \mu^{h(t)} - C_m - 2R - 2A.$$

Using the new constants $K' = KK_\ell K_m$ and $C' = C + C_\ell + C_m + 2R + 2A$, we combine these two equations to get:

$$\frac{1}{K'} \cdot t^k \lambda^t - C' \leq \mu^{h(t)} \leq K' \cdot t^k \lambda^t + C'.$$

Notice that, for fixed $C' > 0$, and $x \geq C'$, we have

$$\log(x + C') \leq \log x + \log 2.$$

When $x < C'$, we have

$$\log(x + C') < \log(2C').$$

Let $M = \max\{\log 2, \log(2C')\}$, so for all x ,

$$\log(x + C') \leq \log x + M.$$

Similarly, for $x > C'$, we have

$$\log(x - C') \leq \log x - M,$$

where $M = \max\{\log 2, \log(2|C'|)\}$. Taking logs yields:

$$-\log K' + k \log t + t \log \lambda - M \leq h(t) \cdot \log \mu \leq \log K' + k \log t + t \log \lambda + M.$$

Thus,

$$|h(t) - \frac{k \log t + t \log \lambda}{\log \mu}| \leq \frac{\log K' + M}{\log \mu}.$$

This shows that

$$h'(t) = \frac{k \log t + t \log \lambda}{\log \mu}$$

is an induced time change function for f with Hausdorff constant $\frac{\log K' + M}{\log \mu}$. \square

Now, we will prove Theorem 33.

Proof. From Theorem 16 (c), we know that G_ϕ has at least one vertical flow line ℓ with a purely exponential divergence rate, say, λ^t . By Theorem 30, $f(\ell)$ is close to some vertical flow line, say m in G_ϕ . By Theorem 16 (b), the divergence rate of m is in \mathcal{C} , that is, it is of the form $t^{\frac{k}{w}} \cdot (\nu^{\frac{1}{w}})^t$. By Lemma 34, there exist c, m such that f has induced time change function

$$h(t) = c \log t + mt.$$

Similar reasoning applies to the coarse inverse of f (denoted f') to show that there are c', m' such that f' has an induced time change

$$h'(t) = c' \log t + m't.$$

Since $f \circ f'$ is a bounded distance from the identity, $h \circ h'$ should also be a bounded distance from the identity. Yet, both $\frac{h(t)}{t}, \frac{h'(t)}{t} \rightarrow \infty$ as $t \rightarrow \infty$, if $k > 0$ or $k' > 0$, respectively. Thus, $\frac{h(h'(t))}{t} \rightarrow \infty$ unless both $c, c' = 0$. This occurs only when $k, k' = 0$. \square

18 Growth Spaces

Definition. Given a Lie group G_ϕ and $\lambda > 0$, define the λ growth subspace as

$$\mathfrak{g}_\lambda = \{v \in \mathfrak{g} \mid \|M^t v\| \preceq \lambda^t t^k \text{ for some } k \in \mathbb{N}\}.$$

This section will be devoted to proving:

Proposition 35 (Growth Spaces are Preserved). *If $f : G_\phi \rightarrow G_\theta$ is a quasi-isometry which preserves the growth space, then for all $\lambda \in \mathbb{R}^\times$*

In order to prove this, we will need the following

Lemma 36 (Growth Spaces are Subalgebras). *For all G_ϕ and $\lambda \in \mathbb{R}^\times$, the λ -growth space $\mathfrak{g}_\lambda \subset \mathfrak{g}$ is a subalgebra.*

Proof. By Proposition 10, any M -invariant subspace is $\text{span}\{e_1, \dots, e_k\}$ for some k . Suppose $e_i, e_j \in \mathfrak{g}_\lambda$. It suffices to prove $[e_i, e_j] \in \mathfrak{g}_\lambda$.

Lemma 17 describes precisely the growth of each component of $M^t e_i$. In the notation given there:

$$\begin{aligned} M^t[e_i, e_j] &= [M^t e_i, M^t e_j] \\ &= [\sum_{l=0}^L \lambda_i^{t-l} t^l e_{i+l}, \sum_{l'=0}^{L'} \lambda_j^{t-l'} t^{l'} e_{j+l'}] \\ &= \lambda_i^t \lambda_j^t \sum_{l=0}^L \sum_{l'=0}^{L'} \lambda_i^{-l} \lambda_j^{-l'} t^{l+l'} [e_{i+l}, e_{j+l'}] \\ &\simeq (\lambda_i \lambda_j)^t \sum_{l=0}^L \sum_{l'=0}^{L'} t^{l+l'} [e_{i+l}, e_{j+l'}]. \end{aligned}$$

The constant implicit in the transition to \simeq is $K = \max\{\lambda_i^{\pm L} \cdot \lambda_j^{\pm L'}\}$.

Proposition 10 (b) implies that, within a Jordan block, weights are nondecreasing. That is, $w_{i+l} \geq w_i$, and $w_{j+l'} \geq w_j$, so $w_{i+l} + w_{j+l'} \geq w_i + w_j$. Since G is Carnot, $[e_{i+k}, e_{j+k}] \in V_{w_i+w_j+l'}$, so

$$\sum_{l=0}^L \sum_{l'=0}^{L'} t^{l+l'} [e_{i+l}, e_{j+l'}] \in V_{w_i+w_j} \oplus \dots \oplus V_k.$$

Furthermore, when this summation is written as a linear combination of the basis elements, the coefficients will all be polynomials in t with degree at most $l + l'$. That is, there are real numbers α_k and real polynomials $p_l(t)$ such that

$$\sum_{l=0}^L \sum_{l'=0}^{L'} t^{l+l'} [e_{i+l}, e_{j+l'}] = \sum_{k=?}^n \alpha_k p_k(t) e_k.$$

Now, apply Corollary 6 to find the nilpotent length of these vectors:

$$||M^t[e_i, e_j]|| \simeq \max_l \{((\lambda_i \lambda_j)^t \alpha_l p_l(t))^{\frac{1}{w_l}}\}.$$

Let L be the value of l which accomplishes the maximum above. Then

$$\begin{aligned} ||M^t[e_i, e_j]|| &\simeq ((\lambda_i \lambda_j)^t p_L(t))^{\frac{1}{w_L}} \\ &\simeq ((\lambda_i \lambda_j)^{\frac{1}{w_L}})^t (p_L(t))^{\frac{1}{w_L}}. \end{aligned} \tag{8}$$

Now turn we direct our attention to the growth rates of e_i and e_j . Assume, without loss of generality, that

$$(\lambda_i)^{\frac{1}{w_i}} \geq (\lambda_j)^{\frac{1}{w_j}}.$$

Then,

$$\begin{aligned} (\lambda_i)^{\frac{w_j}{w_i}} &\geq \lambda_j \\ (\lambda_i)^{\frac{w_i+w_j}{w_i}} &\geq \lambda_j \lambda_i \\ \left((\lambda_i)^{\frac{1}{w_i}}\right)^{\frac{w_i+w_j}{w_k}} &\geq (\lambda_j \lambda_i)^{\frac{1}{w_k}}. \end{aligned}$$

Since $\frac{w_i + w_j}{w_k} \leq 1$, we get that

$$(\lambda_i)^{\frac{1}{w_i}} \geq \left((\lambda_i)^{\frac{1}{w_i}} \right)^{\frac{w_i + w_j}{w_k}} \geq (\lambda_j \lambda_i)^{\frac{1}{w_k}}.$$

By assumption, $\lambda \geq \lambda_i^{\frac{1}{w_i}}$, so $[e_i, e_j] \in \mathfrak{g}_\lambda$. \square

Proof of Proposition 35. By Lemma 36, the growth subspace \mathfrak{g}_λ is a subalgebra. Thus, $\exp(\mathfrak{g}_\lambda)$ is a connected, simply-connected nilpotent Lie group. A powerful theorem of Pansu ([P], Theorem 3 in Section 1) states that if two connected, simply-connected nilpotent Lie groups are quasi-isometric, then the associated graded Lie algebras are isomorphic. In particular, the dimension of each grade is the same. \square

Some Illustrative Examples In each section of this chapter, we present a pair of nilpotent groups which illustrate the potential complexity in classifying such groups via their divergence rates and growth spaces. The first pair of groups have the same permuted absolute Jordan form (and therefore are quasi-isometric) although the quasi-isometry between them is far from being a homomorphism. The second pair of groups agree on the quasi-isometry invariants found in this dissertation but differ in permuted absolute Jordan form. It remains to be determined whether they are quasi-isometric.

Let $N = H \times H \times H$, where H is the Heisenberg group. Thus, N is a two-step nilpotent group with presentation:

$$N = \langle a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9 \mid [a_1, a_2] = a_3, [a_4, a_5] = a_6, [a_7, a_8] = a_9 \rangle.$$

All of the endomorphisms here satisfy the property: If $G < N$ is the infinite cyclic subgroup generated by any one of the nine generators, then ϕ preserves G . That is, for each a_i there is an integer k_i such that $\phi(a_i) = a_i^{k_i}$. Therefore, we can describe the function ϕ by specifying the 9-tuple of exponents. In fact, all of our k_i will be powers of 2, so we prefer to keep track of $n_i = \frac{1}{2} \log_2 k_i$ for $i = 3, 6, 9$ and $n_i = \log_2 k_i$ for $i \neq 3, 6, 9$. The relations of N imply: $n_3 = \frac{n_1 + n_2}{2}$, $n_6 = \frac{n_4 + n_5}{2}$, and $n_9 = \frac{n_7 + n_8}{2}$.

The matrix M representing such an endomorphism is the diagonal matrix with $M_{i,i} = k_i$. Thus, the permuted absolute Jordan form of the matrix is found by permuting the diagonal entries so that k_3, k_6 , and k_9 are the entries in the upper left, in increasing order, followed by $\{k_1, k_2, k_4, k_5, k_7, k_8\}$, also in increasing order.

19 Quasi-isometric Groups Can Be Quite Different

Compare the endomorphism ϕ specified by the 9-tuple

$$(1, 11, 6, 3, 15, 9, 7, 9, 8),$$

with the endomorphism θ specified by the 9-tuple

$$(7, 11, 9, 1, 15, 8, 3, 9, 6).$$

Notice that these multisets are the same. Thus, the sets of divergence rates are also identical. The nilpotent-by-cyclic groups Γ_ϕ and Γ_θ cannot be distinguished via a comparison of divergence rates \mathcal{D} .

Next, we consider the growth spaces associated to these groups. Denote by \mathfrak{g}_n the growth space of Γ_ϕ defined by $\lambda = 2^n$, and similarly denote by \mathfrak{g}'_n the growth space of Γ_θ defined by $\lambda = 2^n$. We denote the Lie algebra of the Heisenberg group by \mathfrak{h} , and the Lie algebra of \mathbb{R}^n by \mathfrak{r}^n . Then,

$$\begin{array}{ll} \mathfrak{g}_1 = \langle a_1 \rangle \cong \mathfrak{r} & \mathfrak{g}'_1 = \langle a_4 \rangle \cong \mathfrak{r} \\ \mathfrak{g}_3 = \langle a_1, a_4 \rangle \cong \mathfrak{r}^2 & \mathfrak{g}'_3 = \langle a_4, a_7 \rangle \cong \mathfrak{r}^2 \\ \mathfrak{g}_6 = \langle a_1, a_4, a_3 \rangle \cong \mathfrak{r}^3 & \mathfrak{g}'_6 = \langle a_4, a_7, a_9 \rangle \cong \mathfrak{r}^3 \\ \mathfrak{g}_7 = \langle a_1, a_4, a_3, a_7 \rangle \cong \mathfrak{r}^4 & \mathfrak{g}'_7 = \langle a_4, a_7, a_9, a_1 \rangle \cong \mathfrak{r}^4 \\ \mathfrak{g}_8 = \langle a_1, a_4, a_3, a_7, a_9 \rangle \cong \mathfrak{r}^5 & \mathfrak{g}'_8 = \langle a_4, a_7, a_9, a_1, a_6 \rangle \cong \mathfrak{r}^5 \\ \mathfrak{g}_9 = \langle a_1, a_4, a_3, a_7, a_9, a_6, a_8 \rangle \cong \mathfrak{h} \times \mathfrak{r}^3 & \mathfrak{g}'_9 = \langle a_4, a_7, a_9, a_1, a_6, a_3, a_8 \rangle \cong \mathfrak{h} \times \mathfrak{r}^3 \\ \mathfrak{g}_{11} = \langle a_1, a_4, a_3, a_7, a_9, a_6, a_8, a_2 \rangle \cong \mathfrak{h}^2 \times \mathfrak{r} & \mathfrak{g}'_{11} = \langle a_4, a_7, a_9, a_1, a_6, a_3, a_8, a_2 \rangle \cong \mathfrak{h}^2 \times \mathfrak{r} \\ \mathfrak{g}_{15} \cong \mathfrak{h}^3 & \mathfrak{g}'_{15} \cong \mathfrak{h}^3 \end{array}$$

Notice that, as we consider the increasing sequence of growth spaces, a copy of \mathfrak{h} is introduced to a growth space whenever we reach the largest of the three eigenvalues associated to the terms of a component group H . For both groups, this occurs at $\lambda = 2^9, 2^{11}$, and 2^{15} .

These growth spaces are all isomorphic. Therefore, they do not tell us that these groups are not quasi-isometric. In fact, these groups have the same permuted absolute Jordan form, and thus are quasi-isometric. Yet, each of the groups is a product of 3 nilpotent-by-cyclic groups, one of the six of which are pairwise quasi-isometric.

20 Another Example

Recall the four-step nilpotent group G described at the end of Section 3 by Equation 1. Compare the nilpotent-by-cyclic groups defined by the two endomorphisms ϕ and θ of G . As in the previous section, each endomorphism acts by raising each generator to a power, which is itself a power of two. Thus, the endomorphism is determined by the growth rates of each generator. This can be calculated for a generator g to be $n = \frac{1}{i} \log_2 k$, where $\phi(g) = g^k$ and $g \in V_i$. For each of the endomorphisms considered here, we have $n_g = 3$ for $g = p, q, r, s, t$. Thus each endomorphism is specified by the

6-tuple $(n_x, n_y, n_z, n_a, n_b, n_c)$. Compare the endomorphism ϕ specified by the 6-tuple

$$(1, 5, 3, 2, 4, 3),$$

with the endomorphism θ specified by the 6-tuple

$$(1, 3, 2, 3, 5, 4).$$

Because these multisets are the same, the divergence rates \mathcal{D}_ϕ and \mathcal{D}_θ also agree. The long but straightforward computation of growth spaces shows that these also agree. However, the matrices associated to these groups do not have the same permuted absolute forms. In fact, these groups are noticeably different. Although the growth rates are the same, they are associated with points at different levels of the group.

A proof that these groups are not quasi-isometric would be further evidence for the conjecture that permuted absolute Jordan form is a quasi-isometry invariant.

The Rigidity of Nilpotent-by-Cyclic Groups

In this chapter, we prove Theorem 1. The proof of the corresponding rigidity theorem for abelian-by-cyclic groups (Theorem 1.2 in [FM00]) proceeds in six steps. Steps 1-3 apply here directly. Step 4 is modified slightly, and Steps 5-6 are not applicable to the nilpotent case. As a result, we have the weaker conclusion that G is (virtually-nilpotent)-by-cyclic, and not the stronger commensurability result found in the abelian-by-cyclic case.

Proof of Theorem 1.

Step 1. The action of G on itself by left multiplication can be conjugated by the quasi-isometry $G \rightarrow X_{N,\phi}$ to give a proper, cobounded quasi-action of G on $X_{N,\phi}$ (see [FM99], Proposition 2.1). Since $[N : \phi(N)] > 1$ we may apply Theorem 26 to conclude that the quasi-action of G on $X_{N,\phi}$ coarsely respects the fibers of the uniform metric fibration $X_{N,\phi} \rightarrow T_{N,\phi}$.

Step 2. Now we use the following result of Mosher, Sageev, and Whyte:

Theorem 37 ([MSW], Theorem 1). *Fix an integer $n \geq 0$ and let Γ be a finite graph of coarse $PD(n)$ groups with bushy Bass-Serre tree. Let H be a finitely-generated group quasi-isometric to $\pi_1\Gamma$. Then H is the fundamental group of a graph of groups with bushy Bass-Serre tree, and with vertex and edge groups quasi-isometric to those of Γ .*

By Step 1, this result applies to the quasi-action of G on $X_{N,\phi}$, because G is quasi-isometric to the finitely-presented group $\Gamma_{N,\phi}$ and so G is finitely presented. The fibers of the map $X_{N,\phi} \rightarrow T_{N,\phi}$ are isometric to N , and it follows that G is the fundamental group of a graph of groups with each vertex and edge group quasi-isometric to nilpotent groups.

Step 3. By Gromov's polynomial growth theorem [G81], any finitely-generated group quasi-isometric to a nilpotent group is *virtually nilpotent*; that is, it has

a finite index nilpotent subgroup. Thus G is the fundamental group of a graph of groups whose vertex and edge groups are virtually nilpotent.

Step 4. Amenability is a quasi-isometry invariant, so since all nilpotent-by-cyclic groups are amenable, G must also be amenable. No amenable group has a nonabelian free subgroup, so G has no free nonabelian subgroup.

Therefore the tree on which G acts has 1 ‘in’ and k ‘out’ branches at each vertex. (If there were two of each, then there would be two translation axes in the group action, and then by the Ping-Pong Lemma, it would have a nonabelian free subgroup). The fact that G acts on a tree of this form implies that G is the ascending HNN extension of some virtually nilpotent group N' ; that is, Γ is (virtually-nilpotent)-by-cyclic. \square

This theorem is false if the condition that Γ be finitely presented is weakened to finitely generated. In fact, Dioubina [D] has found examples of finitely-generated groups which are quasi-isometric to nilpotent-by-cyclic groups, but not themselves (virtually-nilpotent)-by-cyclic.

In particular, she shows that $\mathbb{Z} \wr \mathbb{Z}$ is quasi-isometric to $(Z \oplus F) \wr \mathbb{Z}$, where F is a finite nonsolvable group. However, the former group is solvable, while the latter is not.

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