PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`*. Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2009. Solutions identified as by students are given preference.*

Problems for Solution.

1184. *Arthur L. Holshouser, Charlotte, NC and Benjamin G. Klein, Davidson College, Davidson, NC.*

Suppose that $(S, \odot_1)$ is a group with identity e_1. For s, arbitrary but fixed in S, define a binary operator on S by $a \odot_2 b = a \odot_1 s^{-1} \odot_1 b$ where a and b are elements of S and, for x in S, x^{-1} is the inverse of x in the group $(S, \odot_1)$. (a) Show that $(S, \odot_2)$ is a group and that $(S, \odot_2)$ is isomorphic to the group $(S, \odot_1)$. (b) Express $a \odot_1 b$ in terms of operations in the group $(S, \odot_2)$.

1185. *Proposed by Matthew McMullen, Otterbein College, Westerville, OH.*

Find uncountably many functions, $f_r(x)$, that are positive and continuous on $[1, \infty)$ and that satisfy

$$1 = f_r(1) = \int_1^\infty f_r(x)\,\mathrm{d}x.$$

(Note that $g(x) := 1/x^2$ and $h(x) := e^{-(x-1)}$ are two such functions.)

1186. *Proposed by H. A. ShahAli, Tehran, IRAN.*

Let $a_1, ..., a_n$ be $n \geq 3$ positive reals. Prove that $1 < \sum_{i=1}^{n} \frac{a_i}{a_i + b_i} < n - 1$ for all permutations $(b_1, ..., b_n)$ of $(a_1, ..., a_n)$, and that 1 and $n - 1$ are the best possible.

1187. *Proposed by Brian Bradie, Christopher Newport University.*

Evaluate

$$\int_0^1 \frac{\ln(1+x)}{x}\,dx.$$

1188. *Proposed by Javier Gomez-Calderon and David Wells, Penn State University at New Kensington.*

Find all real polynomials P having the property that $P(x-1)P(x) = P(x^2)$ for all x.

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1189. *Proposed by Peter A. Lindstrom, Batavia, NY.*

If F_n denotes the nth Fibonacci number, show that $F_{4n+2} - (2n+1)$ is divisible by 5.

1190. *Proposed by Tom Moore, Bridgewater State College, Bridgewater, MA.*

Prove that every even perfect number is both a sum and a difference of two distinct deficient numbers.

1191. *Proposed by Fred Weber, Lyndhurst, OH.*

For integers $m \geq 2$ and $d \geq 0$, let $\phi(d, m)$ be the number of ordered pairs $< a, b >$ of residues modulo m for which $b = a + d$ and both a and b are relatively prime to m. Find a formula for $\phi(d, m)$.

1192. *Proposed by Scott D. Kominers, student, Harvard University, Cambridge, MA.*

For which positive integers n is it true that, for all partitions P of n into more than $\lfloor \frac{n}{2} \rfloor$ parts, there is an undirected graph G with vertex set $V(G)$ such that $\deg(V(G)) = P$?

1193. *Proposed by Arthur L. Holshouser, Charlotte, NC.*

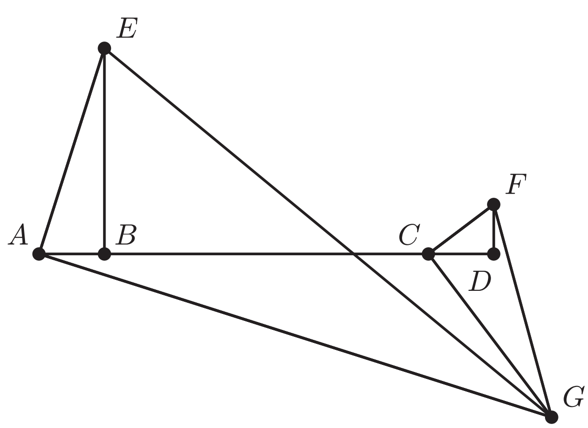
Suppose f is a reasonably well-behaved real function $f(x)$ that satisfies for all real numbers a, b the condition

$$f(a+b) = \frac{f(a) + f(b)}{1 - f(a) f(b)}.$$

Prove that $f(x) = \tan(mx)$ where m is any fixed real number.

1194. *Proposed by Herman J. Servatius, Worcester Polytechnic Institute, Worcester, MA.*

Given seven points $\{A, \ldots G\}$ in the plane such that A, B, C, and D are colinear, $\overline{AB} = \overline{CD}$, $AB \perp BE$, $CD \perp DF$, $AE \perp AG$, and $CF \perp CG$.



1. Prove that if the four colinear points occur in the order (A, B, C, D) then $\triangle ABE \sim \triangle CDF$.
2. For what other orderings of the four points does that conclusion hold?

1195. *Proposed by Mike Pinter, Belmont University, Nashville, TN.*

Consider the following alphametic: $M\,C\,C\,A\,I\,N + O\,B\,A\,M\,A\ =\ D\,E\,C\,I\,D\,E$
As we cast our vote, we want to maximize our decision. Find the maximum value of DECIDE for the alphametic.