PROBLEM DEPARTMENT

ASHLEY AHLIN AND HAROLD REITER*

This department welcomes problems believed to be new and at a level appropriate for the readers of this journal. Old problems displaying novel and elegant methods of solution are also invited. Proposals should be accompanied by solutions if available and by any information that will assist the editor. An asterisk (*) preceding a problem number indicates that the proposer did not submit a solution.

All correspondence should be addressed to Harold Reiter, Department of Mathematics, University of North Carolina Charlotte, 9201 University City Boulevard, Charlotte, NC 28223-0001 or sent by email to `hbreiter@uncc.edu`. *Electronic submissions using LaTeX are encouraged. Other electronic submissions are also encouraged. Please submit each proposal and solution preferably typed or clearly written on a separate sheet (one side only) properly identified with name, affiliation, and address. Solutions to problems in this issue should be mailed to arrive by March 1, 2013. Solutions identified as by students are given preference.*

Problems for Solution.

1264. *Proposed by Peter Linnell, Vinginia Polytechnic University, Blacksburg, VA.* This was problem 7 on the annual VPI Regional College Math Contest, 1997. The VTRC bus company serves cities in the USA. A subset $\mathcal{S}$ of the cities is called well-served if it has at least three cities and from every city $\mathcal{A}$ in $\mathcal{S}$, one can take a nonstop VTRC bus to at least two different other cities $\mathcal{B}$ and $\mathcal{C}$ in $\mathcal{S}$ (though there is not necessarily a nonstop VTRC bus from $\mathcal{B}$ to $\mathcal{A}$ or from $\mathcal{C}$ to $\mathcal{A}$). Suppose there is a well-served subset $\mathcal{S}$. Prove that there is a well-served subset $\mathcal{T}$ such that for any two cities $\mathcal{A}, \mathcal{B}$ in $\mathcal{T}$, one can travel by VTRC bus from $\mathcal{A}$ to $\mathcal{B}$, stopping only at cities in $\mathcal{T}$.

1265. *Proposed by Mihaela Blanariu, Columbia College, Chicago, IL.*

For each nonnegative integer n, let

$$x_n = \frac{1}{(b-a)^{2n+1}} \int_a^b (x-a)^n (b-x)^n dx,\ a < b.$$

Show that $\sum_{n=1}^{\infty} x_n$ is convergent.

1266. *Proposed by Thomas Moore, Bridgewater State University, Bridgewater, MA.*

In 1770 Lagrange proved that every positive integer is the sum of at most four squares of nonnegative integers. In particular, the pentagonal numbers $1, 5, 12, 22, \ldots$ which are defined by $P_n = n(3n-1)/2$, $1, 2, 3, \ldots$, have such a representation. Looking more closely we ask you to prove that there are infinitely many pentagonal numbers that are not the sum of three squares of positive integers.

1267. *Proposed by Arthur Holshouser, Charlotte, NC*

If m_1, m_2, and m_3 are real numbers find values of ϕ, x, y, z satisfying the four conditions

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1. $\frac{\phi(\frac{1}{x}+y+1)}{1+\phi} = m_1$,
2. $\frac{\phi(\frac{1}{z}+x+1)}{1+\phi} = m_2$,
3. $\frac{\phi(\frac{1}{y}+z+1)}{1+\phi} = m_3$, and
4. $xyz = 1$.

1268. *Proposed by Gurshamnjot Singh, student*, University of California, Berkeley.

Suppose the polynomial $p(x) = ax^3 + bx + c$ has a single real zero d. Given that $a > 0$ and $b < 0$, show that $d \notin \left[-2\sqrt{\frac{-b}{3a}},\ 2\sqrt{\frac{-b}{3a}} \right]$.

1269. *Proposed by Mike Pinter, Belmont University, Nashville, TN.*

As our U.S. Presidential race heats up, consider the following Base 10 alphametic:

$$\begin{array}{r} R\ O\ M\ N\ E\ Y \\ \underline{+\ O\ B\ A\ M\ A} \\ B\ A\ T\ T\ L\ E \end{array}$$

Find a solution to the alphametic.

1270. *Proposed by Ben Klein, Davidson College, Davidson, NC.*

Suppose that f is a function such that $f'(x) > 0$ and $f''(x) > 0$ for x in the interval $[a, b]$.

(a) Explain why a trapezoidal approximation to $\int_a^b f(x)\ dx$ must provide an overestimate for the integral.

(b) Suppose that $a \le x_1 \le b$ and that a trapezoidal approximation for $\int_a^b f(x)\ dx$ is computed using the subintervals $[a, x_1]$ and $[x_1, b]$. Show that the value of x_1 that produces the approximation that is closest to the actual value of the definite integral satisfies the equation:

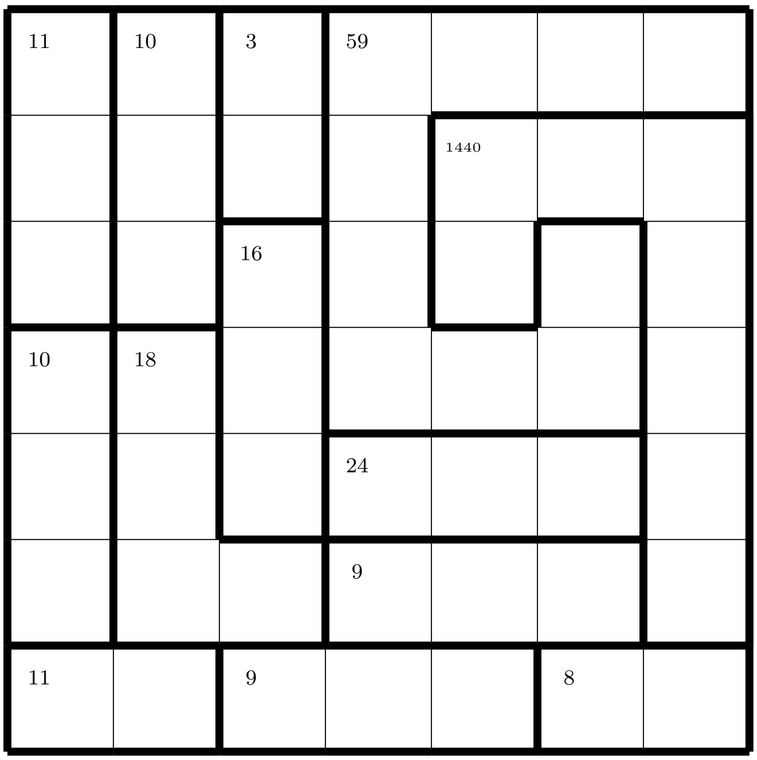
$$f'(x_1) = \frac{f(b) - f(a)}{b - a}.$$

(c) Apply the result from part (b) to the function $f(x) = x/(2-x)$ on the interval $[0, 1]$. .

(d) For the function f given in part (c), find the values of x_1 and x_2 such that the trapezoidal approximation for $\int_a^b f(x)\ dx$ that uses the subintervals $[0, x_1]$, $[x_1, x_2]$ and $[x_2, b]$ is closest to the actual value of the integral.

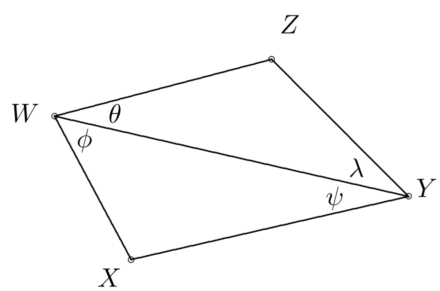
1271. *Proposed by Dave Petranick, Charlotte, NC.*

In the standard puzzle KenKen® the numbers in each heavily outlined set of squares, called *cages*, must combine (in any order) to produce the *target number* in the top corner of the cage using one of the four mathematical operations $+, \times, -, \div$. Of course, each cage with more than two cells must be an additive cage or a multiplicative cage. A number can be repeated within a cage as long as it is not in the same row or column. In this 7×7 puzzle, the seven numbers are 1 through 7. Each clue can be assigned an operation so that the resulting standard KenKen puzzle has at least one solution. Find, with proof, the location of all the sevens.

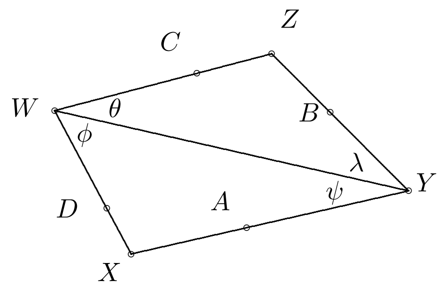


1272. *Proposed by Stephen West, SUNY Geneseo; Stephen Greenberg, East Amherst, NY; and Arthur Holshouser, Charlotte, NC.*

Suppose $WXYZ$ is a convex quadrilateral having a known shape; that is, the four angles $\theta, \phi, \psi, \lambda$ are known. However, suppose we do not know the four points W, X, Y, Z.



Suppose also that point A belongs to side XY, the point B belongs to side YZ, the point C belongs to side ZW and the point D belongs to side WX. The four points A, B, C, D are all given. Show how to construct the four points W, X, Y, Z from the given four points A, B, C, D.



1273. *Proposed by Arthur Holshouser, Charlotte, NC.*

If $x\dot{+}y = \frac{x+y}{xy+1}$, show that the following is true.

1. $((-1,1),0,\dot{+})$ is an Abelian group on $(-1,1)$ with identity 0.
2. Every continuous automorphism $f : ((-1,1),0,\dot{+}) \to ((-1,1),0,\dot{+})$ can be expressed as $f_r(x) = \frac{(1+x)^r-(1-x)^r}{(1+x)^r+(1-x)^r}$ for some non-zero real number r.