December 14, 2006   Your name   _____

Throughout this test you must **show your work**. Answers without supporting work are generally worth about one fourth credit. The total number of points available on this test is 215.

1. (15 points) Find the base 9 representation of the **decimal** 2007. Then interpret your base nine numeral as a sum of multiples of powers of 9 to get the decimal representation of your number.

2. (15 points) Find nonzero digits a, b, c, and d such that $343 \cdot a + 49 \cdot b + 7 \cdot c + d = 2007$. Hint: Is the left side a sum of multiples of powers of 7?

3. (15 points) You are given two decanters, one of capacity 7 units and the other of capacity 10 units.

   (a) How can you measure exactly 1 unit of liquid?

   (b) How can you measure exactly 9 units of liquid?

4. (15 points) Pick any 3 nonzero digits, like 2,4, and 9. Construct the six-digit number $249,249$ and factor it into primes: $249,249 = 3 \cdot 7 \cdot 11 \cdot 13 \cdot 83$.

   (a) Try this twice with two different numbers, each time factoring your six-digit number. Be sure to pick something different from my pick. What prime numbers occur in all three (your two and mine) of these factorizations.

   (b) Show that for any three nonzero digits a, b, and c, the six-digit number $\underline{abcabc}$ is a multiple of 7 or find an example of a number of this form that is not a multiple of 7.

5. (15 points) **Finding the unknown digit.**
   Let $N = \underline{abcde}$ denote the five digit number with digits a, b, c, d, e and $a \neq 0$. Let $N' = \underline{edcba}$ denote the reverse of N. Suppose that $N > N'$ and that $N - N' = 670x3$ where x is a digit. What is x?

6. (20 points) Recall the subtraction game in which two players start with some positive integers written on a board. The first player subtracts one of the numbers on the board from a larger one on the board, and writes down the **new** difference. At each stage, the next player finds a positive difference between two numbers on the board that is not already written on the board and writes it on the board. The first player who cannot find a new positive difference loses. For each of the sets of numbers listed below, decide how many numbers will be on the board at the end of the game. Use this information to state whether you would like to move first or not (in order to win). For each part, 1 point for the correct decision and 5 points for the correct explanation.

   (a) 3105 and 4104

   (b) $21, 24, 81, 87$

   (c) $101, 102, 103$

7. (20 points) Rational and irrational numbers.

   (a) Is the number $M = 0.0123456789101112\ldots$ rational or irrational? Elaborate on your answer.

   (b) Is the number $N = 0.0123123\overline{123}$ rational or irrational? Elaborate on your answer.

8. (20 points) Consider the number $M = 0.0123456789101112\ldots$ obtained by writing the nonnegative integers in order next to one another after the decimal point. In this problem all numbers are in their usual decimal notation. The first 3 parts seem to have nothing to do with the number M. Their purpose is to help you with parts (d), (e) and (f).

   (a) How many single-digit positive integers are there?

   (b) How many two-digit positive integers are there?

   (c) How many three-digit positive integers are there?

   (d) What is the 26th digit to the right of the decimal point of M?

   (e) What is the 206th digit to the right of the decimal point of M?

   (f) What is the 2006th digit to the right of the decimal point of M?

9. (40 points) We have investigated three major theorems in this course. Pick out one of them and discuss both its proof and its important implications.

10. (20 points) Find the number of elements of each of the lists of numbers below. Note that item (a) could be phrased 'how many 4-digit multiples of 7 are there?'

   (a) $1001, 1008, 1015, 1022, 1029, \ldots, 9996$

   (b) $1001, 1012, 1023, 1034, 1045, \ldots, 9999$

   (c) $1001, 1014, 1027, 1040, 1054, \ldots, 9997$

   (d) $113, 118, 123, 128, 133, \ldots, 438$

   (e) $48, 55, 62, 69, 76, \ldots, 699$

   (f) $109, 113, 117, 121, 125, \ldots, 201$

11. (20 points) Take another (different from the theorem discussed above) major IDEA covered in the course. Discuss the idea and its implications.