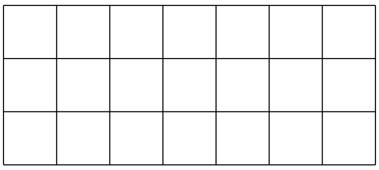
Pigeon Hole Principle Problems

1. Is it true that from any 30 different natural numbers, not greater than 50 one can choose a pair such that one number in the pair is twice the other.

   Solution: The answer is 'false' for the following reason. We can include the smaller member from each of the following sets: $\{1,2\}, \{4,8\}, \{16,32\}$, $\{3,6\}, \{12,24\}, \{5,10\}, \{7,14\}, \{9,18\}, \{11,22\}, \{13,26\}, \{15,30\}, \{19,38\}$, $\{20,40\}, \{21,42\}, \{44\}, \{23,46\}, \{24,48\}, \{25,50\}, \{17\}, \{27\}, \{29\}, \{31\}$, $\{33\}, \{35\}, \{37\}, \{39\}$, $\{41\}, \{43\}, \{45\}, \{47\}, \{49\}$. Another way to see this is to build a list. Each time a number is included, its double has to be removed. I appreciate Linda Chandler's idea: 1, 3, 4, 5, 7, 9, 11, 12, 13, 15, 16, 17, 19, 20, 21, 23, 25, 27, 28, 29, 31, 33, 35, 37, 39, 41, 43, 45, 47, 49, a list of 30 different numbers.

2. If each square of a 3-by-7 chessboard is colored either black or white, then the board must contain a rectangle consisting of at least four squares whose corner squares are either all white or all black.

   Show that if the grid is only 3-by-6, there are colorings for which the conclusion fails.

   Solution: We examine the columns of the grid. If one column is all white, then we'll be done if any other column has at least two white squares. But there are only four possible columns with one or no white squares. Therefore, either we must include a column that has at least two white squares, or we have to duplicate a column. If the same pattern is duplicated, then these two columns have two squares each on the same color.

   If there is an all black column, the reasoning can be duplicated. Therefore we need only consider the case when there is no entirely black column and no entirely white column. In this case, there are just six possible patterns, BBW, BWB, WBB, BWW, WBW, and WWB. But there are seven columns to color, so one of these patterns must be duplicated. We saw above that when a pattern is duplicated, a rectangle can be found all of whose are the same color.

3. Prove that among any set of 51 positive integers less than 100, there is a pair whose sum is 100.

Solution: Let $a_1, a_2, \ldots, a_{51}$ denote these 51 positive integers, Let $S_1 = \{1, 99\}, S_2 = \{2, 98\}, S_3 = \{3, 97\}, \ldots, S_k = \{k, 100 - k\}, \ldots, S_{49} = \{49, 51\}$, and $S_{50} = \{50\}$. Let $S_1, S_2, \ldots, s_{50}$ denote the pigeonholes, and $a_1, a_2, \ldots, a_{51}$ the pigeons. By PHP, we must have at least one S_i with two numbers in it. Their sum is a 100.

4. Fifteen children together gathered 100 nuts. Prove that some pair of children gathered the same number of nuts.

   Solution: Proof by contradiction. Suppose all the children gathered a different number of nuts. Then the fewest total number is $0 + 1 + 2 + 3 + 4 + 5 + 6 + \cdots + 14 = 105$, but this more than 100, a contradiction.

5. The integers from 1 to 10 are randomly distributed around a circle. Prove that there must be three neighbors whose sum is at least 17. What about 18? What about 19?

   Solution: Here's a summary of the proof we saw in class for the number 17. Name the numbers in clockwise order starting somewhere. Let the numbers be named $a_1, a_2, a_3, \ldots, a_{10}$. Let $S_1 = a_1 + a_2 + a_3$, etc, out to $S_{10} = a_{10} + a_1 + a_2$. We get 10 equations,

$$\begin{aligned}
S_1 &= a_1 + a_2 + a_3 \\
S_2 &= a_2 + a_3 + a_4 \\
S_3 &= a_3 + a_4 + a_5 \\
S_4 &= a_4 + a_5 + a_6 \\
S_5 &= a_5 + a_6 + a_7 \\
S_6 &= a_6 + a_7 + a_8 \\
S_7 &= a_7 + a_8 + a_9 \\
S_8 &= a_8 + a_9 + a_{10} \\
S_9 &= a_9 + a_{10} + a_1 \\
S_{10} &= a_{10} + a_1 + a_2
\end{aligned}$$

   The sum of the 10 numbers $a_1, \ldots, a_{10}$ is 55 because the ten numbers are, in some order, just the numbers 1 through 10. So the sum of all the as on the right is 165. In other words $S_1 + S_2 + \cdots + S_{10} = 165$. This means that the average value of the 10 S_is is 16.5. Therefore at least one of them is at least 17. To prove that 18 is always achieved, find the a_i that is 1, and break the other nine numbers into three groups such that the groups don't overlap. These nine (none 1) numbers have sum 54, so the average sum of three is 18. At least one member in a group must at least average, so the number 18 must be achieved. To see that 19 need not be achieved, you can arrange the numbers clockwise around the circle as follows: $1, 10, 6, 2, 9, 5, 4, 8, 3, 7$. Note that the sum 18 is achieved four times, with the sums $10 + 6 + 2$, $9 + 5 + 4$ and $8 + 3 + 7$.

6. There are 33 students in the class and sum of their ages 430 years. Is it true that one can find 20 students in the class such that sum of their ages greater 260?

   Solution: Give a name to the ages. Call them $a_1, a_2, a_3, \ldots, a_{33}$ and assume $a_1 \leq a_2 \leq a_3 \leq \cdots \leq a_{33}$. Now suppose for the moment that the 20 largest of these, $a_{14}, a_{15}, \ldots, a_{33}$ have a sum that is at most 260. Since the average age of these 20 students is 13, it follows that $a_{14} \leq 13$. This means each of the numbers $a_1, a_2, \ldots, a_1 3$ is at most 13. Adding all these numbers together, we get $a_1 + a_2 + a_3 + \cdots + a_{33} \leq a_1 + a_2 + a_3 + \cdots + a_{13} + 260 \leq 169 + 260 = 429$, a contradiction.