November 16, 2006   Your name   _____
Throughout this test you must **show your work**. Answers without supporting work
are generally worth about one fourth credit. The total number of points available
on this test is 141.

1. (12 points) Make up a four-digit number N that has all different digits.

   (a) Find the base 6 representation of N.

   (b) Interpret your answer to get the number that you started with.

2. (10 points) Find a pair of positive irrational numbers whose sum is 1.

3. (12 points) Card Trick Problem

   Carry out the assistants job on the card trick by arranging the five cards given in the following order. The first card is the one to be hidden, the second is the anchor, and the last three are arranged in the order that determines the number of steps to take to get from the anchor card to the hidden card.

   (a) $K\clubsuit, 9\diamondsuit, 6\heartsuit, J\heartsuit, J\spadesuit$

   (b) $7\clubsuit, 7\diamondsuit, 7\heartsuit, 6\diamondsuit, 7\spadesuit$

   (c) $J\clubsuit, Q\diamondsuit, K\heartsuit, A\spadesuit, 8\spadesuit$

4. (15 points) Rational representation.

   (a) Use long division to find the decimal representation of $\frac{17}{27}$.

   (b) Use the 'kill the tail' algorithm discussed in class to find a pair of integers m, n such that m/n is the decimal you found in part (a).

   (c) Reduce to lowest terms the fraction you found in part (b).

5. (20 points) For each of the following positions, find a winning move if there is one, and state otherwise if there is not one. Recall that, for example, $(75, 20)$ refers to the position with 75 counters and maximum move of size 20.

(a) In all five games below, you're playing $N_i(k)$, that is, one pile dynamic identity nim in which each move must be no bigger than the previous move.

   i. $(74, 20)$

   ii. $(84, 20)$

   iii. $(64, 20)$

   iv. $(144, 15)$

   v. $(144, 25)$

(b) In all five games below, you're playing $N_d(k)$, that is, one pile dynamic doubling nim in which each move must be no bigger than twice previous move.

   i. $(75, 20)$

   ii. $(84, 20)$

   iii. $(64, 20)$

   iv. $(178, 30)$

   v. $(233, 89)$

6. (12 points) Notice that $7123 - 3127 = 3996 = 2^2 \cdot 27 \cdot 37$. Also notice that $8223 - 3228 = 4995 = 5 \cdot 27 \cdot 37$.

   (a) Is it true that for any four digits, a, b, c and d, $\underline{abcd} - \underline{dbca}$ is a multiple of 27? Explain why or why not.

   (b) Is it true that for any four digits, a, b, c and d, $\underline{abcd} - \underline{dbca}$ is a multiple of 37? Explain why or why not.

7. (12 points) Find the units digit of the Fibonacci Number F_{2006}

8. (12 points) Is there a real number between $0.12\overline{3}$ and 0.124? If so, give an example of such a number; if not, just say "no."

9. (12 points) If the number M is an irrational number, then $1/M$ must be an irrational number as well. True or False? Give a reason for your answer. (2 points for correct answer, 10 points for the reasoning.)

10. (12 points) Use the Euclidean algorithm to solve the decanting problem for decanters of sizes 215 and 219. In other words, find integers x and y such that $gcd(215, 219) = 215x + 219y$. Then explain how this solves the decanting problem.

11. (12 points) Notice that $(-5)^2 = 25$, $3^2 = 9$ and $7^2 = 49$ are all 1 bigger than a multiple of four. That is, all three of 25, 9 and 49 yield a reminder of 1 when divided by 4. Another way to say this is $(-5)^2 \equiv 1 \pmod 4$, $3^2 \equiv 1 \pmod 4$, and $7^2 \equiv 1 \pmod 4$. Pick two other numbers, like $-5, 3$ and 7 that are three bigger than a multiple of 4, square them and see if the result is congruent to 1 modulo 4. Next prove that for any integer n that is congruent to 3 modulo 4, n^2 is congruent to 1 modulo 4.