May 4, 2001 **Name** ____________________

The first five problems count 7 points each (total 35 points) and rest count as marked. There are 195 points available. Good luck.

1. Consider the function f defined by:

$$f(x) = \begin{cases} 2x^2 - 3 & \text{if } x < 0 \\ 5x - 3 & \text{if } x \geq 0 \end{cases}$$

Find the slope of the line which goes through the points $(-2, f(-2))$ and $(3, f(3))$.

(A) $7/5$ **(B)** 2 **(C)** $17/5$ **(D)** 5 **(E)** 7

Solution: The two points on the graph are $(-2, 5)$ and $(3, 12)$ and the slope of the line joining them is $m = 7/5$.

2. The distance between the point $(6.5, 8.5)$ and the midpoint of the segment joining the points $(2, 3)$ and $(5, 6)$ is

(A) $\sqrt{22}$ **(B)** $\sqrt{23}$ **(C)** 5 **(D)** $\sqrt{26}$ **(E)** 6

Solution: The midpoint of the segment is $3.5, 4.5)$, so the distance is $d = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$.

3. Let $f(x) = 2x + 3$ and $g(x) = 3x - 9$. Which of the following does not belong to the domain of f/g?

(A) 1 **(B)** 3 **(C)** 6 **(D)** 9 **(E)** The domain of f/g is the set of all real numbers.

Solution: Only a number for which g is zero fails to be in the domain. Solving $3x - 9 = 0$ yields $x = 3$.

4. The line tangent to the graph of a function f at the point $(2, 5)$ on the graph also goes through the point $(0, 7)$. What is $f'(2)$?

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Solution: The slope of the line through $(2, 5)$ and $(0, 7)$ is -1.

5. What is the slope of the tangent line to the graph of $f(x) = x^{-2}$ at the point (2,1/4)?

(A) $-1/4$ **(B)** $-1/8$ **(C)** $-1/16$ **(D)** $-1/256$ **(E)** $-1/512$

Solution: The derivative is $f'(x) = -2x^{-3}$ whose value of at $x = 2$ is $f'(2) = -1/4$.

6. (15 points) Let $f(x) = 1/(3x)$.

   (a) Construct $\frac{f(2+h)-f(2)}{h}$

   Solution: $\frac{f(2+h)-f(2)}{h} = \frac{\frac{1}{3(2+h)}-\frac{1}{6}}{h} = -\frac{1}{(2+h)\cdot 6}$.

   (b) Simplify and take the limit of the expression in (a) as h approaches 0 to find $f'(2)$.

   Solution: $\lim_{h\to 0} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0} -\frac{1}{(2+h)\cdot 6} = -1/12$.

   (c) Use the information found in (b) to find an equation for the line tangent to the graph of f at the point $(2, 1/6)$.

   Solution: $y - 1/6 = -(1/12)(x-2)$.

7. (10 points) Find the rate of change of $f(t) = e^{2t} \cdot \ln(t)$ when $t = 1$.

   Solution: Use the product rule to get $f'(t) = 2e^{2t} \cdot \ln(t) + (1/t) \cdot e^{2t}$ whose value at $t = 1$ is $f'(1) = 2e^2 \cdot \ln(1) + (1/1) \cdot e^2 = e^2$.

8. (20 points) Suppose the functions f and g are differentiable and their values at certain points are given in the table. The next four problems refer to these functions f and g. Notice that, for example, the entry 1 in the first row and third column means that $f'(0) = 1$. Note also that, for example, if $K(x) = f(x) - g(x)$, then $K'(x) = f'(x) - g'(x)$ and $K'(4) = f'(4) - g'(4) = 5 - 10 = -5$. Answer each of the questions below about functions that can be build using f and g.

x	$f(x)$	$f'(x)$
0	2	1
1	2	3
2	5	4
3	1	2
4	3	5
5	6	4
6	0	5
7	4	1

x	$g(x)$	$g'(x)$
0	5	5
1	7	3
2	4	6
3	2	6
4	6	10
5	3	3
6	1	2
7	0	1

   (a) The function h is defined by $h(x) = f(g(x))$. Use the chain rule to find $h'(3)$.

   Solution: By the chain rule, $h'(3) = f'(g(3) \cdot g'(3) = f'(2) \cdot g'(3) = 4 \cdot 6 = 24$.

(b) The function k is defined by $k(x) = f(x) \cdot g(x)$. Use the product rule to find $k'(1)$.

Solution: $k'(1) = f'(1) \cdot g(1) + f(1) \cdot g'(1) = 3 \cdot 7 + 2 \cdot 3 = 27.$

(c) The function H is defined by $H(x) = f(f(x))$. Use the chain rule to find $H'(2)$.

Solution: $H'(2) = f'(f(2)) \cdot f'(2) = f'(5) \cdot f'(2) = 4 \cdot 4 = 16.$

(d) Let $Q(x) = f(f(x) - g(x))$. Find $Q'(5)$.

Solution: $Q'(5) = f'(f(5)-g(5)) \cdot (f'(5)-g'(5)) = f'(6-3) \cdot (4-3) = 2.$

9. (10 points) A radioactive substance has a half-life of 27 years. Find an expression for the amount of the substance at time t if 20 grams were present initially.

   Solution: $Q(t) = Ae^{-kt}$. Since the half-life is 27 years, it follows that $.5 = e^{-27k}$, which can be solved to give $k \approx 0.0025672$. Thus $Q(t) = 20e^{-0.0025672t}$.

10. (10 points) If $h = g \circ f$ and $f(1) = 2, g'(2) = 5, f'(1) = -3$ find $h'(1)$.

    Solution: $h'(1) = g'(f(1)) \cdot f'(1) = g'(2) \cdot f'(1) = -15$.

11. (15 points) Let $f(x) = x^4 + 2x^3 - 6x^2 + x - 5$.

    (a) Find the interval(s) where f is concave upward.
    Solution: $f'(x) = 4x^3 + 6x^2 - 12x + 1$ and $f''(x) = 12x^2 + 12x - 12$, which has two zeros, $x = (-1 \pm \sqrt{5})/2$. So f'' is positive over the intervals $(-\infty, (-1-\sqrt{5})/2$ and $(-1+\sqrt{5})/2, \infty)$.

    (b) Find the inflection points of f, if there are any.
    Solution: There are two inflection points, $(-1+\sqrt{5})/2, -6.0556)$ and $(-1-\sqrt{5})/2, f(-1-\sqrt{5})/2) = (-1-\sqrt{5})/2, -23.944)$

12. (20 points) A ball is thrown upwards from the top of a building that is 200 feet tall. The position of the ball at time t is given by $s(t) = -16t^2 + 36t + 200$, where $s(t)$ is measured in feet and t is measured in seconds.

    (a) What is the velocity of the ball at time $t = 0$?
    Solution: $s'(t) = -32t + 36$ and $s'(0) = 36$.

    (b) What is the velocity of the ball at time $t = 1$?
    Solution: $s'(1) = -32 \cdot 1 + 36 = 4$.

    (c) How many seconds elapse before the ball hits the ground?
    Solution: Solve $-16t^2 + 36t + 200 = 0$ to get $t \approx 4.83$.

    (d) What is the speed of the ball when it hits the ground?
    Solution: $s'(4.83) \approx -118.72$.

    (e) What is the acceleration of the ball at the time it hits the ground?
    Solution: $a(t) = v'(t) = s''(t) = -32ft/sec^2$.

13. (20 points)

    (a) Let $f(x) = 2x^2$ and compute the Riemann sum of f over the interval $[1, 9]$ using four subintervals of equal length ($n = 4$) and choosing the representative point in each subinterval to be the midpoint of the subinterval.

    Solution: The endpoints of the intervals are $2, 4, 6, 8$ and the sum in question is $f(2) \cdot (3-1) + f(4) \cdot (5-3) + f(6) \cdot (7-5) + f(8) \cdot (9-7) = 2(8 + 32 + 72 + 128) = 480$.

    (b) Compute
    $$\int_1^9 2x^2\ dx$$
    and compare this value with the one in part a.

    Solution: $\int_1^9 2x^2\ dx = \frac{2x^3}{3}|_1^9 = \frac{2}{3}9^3 - \frac{2}{3}1^3 = 485\frac{1}{3}$.

14. (10 points) Find an equation for the line tangent to the graph of $f(x) = x\ln(x) - x$ at the point $(1, f(1))$.

    Solution: $f'(x) = \ln(x) + x \cdot frac1x - 1$ so $f'(1) = 0 + 1 \cdot 1 - 1 = 0$, and since $f(1) = -1$, it follows that the tangent line has the equation $y = -1$.

15. (10 points) Evaluate $\int 3x^2\sqrt{x^3+1}\ dx$

    Solution: Let $u = x^3 + 1$. Then $du = 3x^2dx$ and $\int 3x^2\sqrt{x^3+1}\ dx = \frac{2}{3}(x^3 + 1)^{3/2} + C$.

16. (10 points) Evaluate $\int_1^3 x^3 \cdot (x^4-2)^2\ dx$

    Solution: Let $u = x^4 - 2$. Then $du = 4x^3dx$ and $\int_1^3 x^3 \cdot (x^4-2)^2\ dx = \frac{1}{4}(x^4-2)^3|_1^3 = \frac{79^3}{12} - \frac{-1}{12} = 41086.5$.

17. (10 points) Evaluate $\int_0^4 2xe^{x^2}\ dx$

    Solution: $\int_0^4 2xe^{x^2}\ dx = e^{x^2}|_0^4 = e^{16} = e^0 \approx 8886110.5 - 1 = 8886109.5$.