June 26, 2001 Your name ____________________

The multiple choice problems count five points each. The total value of this test is 215 points.

1. Let $f(x) = \ln(x^4)$. What is $f'(e^2)$?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** $4e^{-2}$ **(E)** $4e^2$

   Solution: D. Note that $f'(x) = 4x^3/x^4 = 4/x$, so $f'(e^2) = 4/e^2 = 4e^{-2}$.

2. Which of the following is closest to a solution to $2e^{x^2+1} = 2001$?

   (A) 1.74 **(B)** 2.43 **(C)** 6.91 **(D)** 10 **(E)** 31.59

   Solution: B. Divide by 2 and then take natural log of both sides to get $\ln(e^{x^2+1} = \ln(2001/2)$. Thus $x^2 + 1 \approx 6.9082$. Subtract 1 and take squares roots to get $x = \pm\sqrt{6.9082 - 1} \approx 2.43$ when we use the + sign.

3. For how many values of x is it true that

   $$\log_3[(x^2 - 9)(x^2 - 4) + 3] = 1?$$

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: E. The exponential form is $3^1 = (x^2 - 9)(x^2 - 4) + 3$, which is equivalent to $(x^2 - 9)(x^2 - 4) = 0$, which has the four roots, $x = \pm 3, x = \pm 2$.

4. Which of the following is closest to a solution of $\log_7(x + 1) + \log_7(x + 2) = \log_7 12$

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

   Solution: A. The equation is equivalent to $\log_7(x + 1)(x + 2) = \log_7 12$ from which it follows that $(x + 1)(x + 2) = 12$. Solve this to get $x = 2$ and $x = -5$, the later of which makes no sense.

From here on, it is important that you **show your work**.

5. (10 points) The doctor has told Mr. Tobigwaiste that he can expect a weight loss of 2% per week during the ten weeks of treatments he is about to begin. If his weight before the treatments begin is 244 pounds, what is his weight expected to be, to the nearest pound at the end of the ten weeks of treatments?

   Solution: This is the exponential decay model. Let $W(t) = (0.98)^t \cdot 244$ represent the weight function where t is measured in weeks. Then, $W(10) = .98^{10} \cdot 244 \approx 199.37$.

6. (10 points) A wolf population has decreased to 50 wolves but growth has begun at a rate of 1.2% per year. How many wolves, to the nearest whole number, can be expected in 10 years?

   Solution: This is the exponential growth model. Let $F(t) = (1.02)^t \cdot 50$ represent the population function where t is measured in years. Then, $W(10) = 1.02^{10} \cdot 50 \approx 1.219 \cdot 50 \approx 61$ wolves.

7. (10 points) Find an equation for the line tangent to the graph of $f(x) = e^{x^2+1}$ at the point $(1, e^2)$

   Solution: The derivative is $f'(x) = 2xe^{x^2+1}$, whose value at $x = 1$ is $2e^2$. Thus the tangent line is the horizontal line $y - e^2 = 2e^2(x-1)$, that is $y = 2e^2x - e^2$.

8. (10 points) Solve each of the equations below for x in terms of the other letters.

   (a) $6 \cdot 2^{3x} = 18$
   Solution: Divide by 6 to get $2^{3x} = 3$, so $3x \log 2 = \log 3$, so $x = \frac{\log 3}{3 \log 2} \approx 0.528$

   (b) $\frac{1}{1+2^x} = \frac{2}{66}$.
   Solution: Take reciprocals to get $1 + 2^x = 33$ from which it follows that $2^x = 32$ and $x = 5$.

9. (30 points) Suppose that the derivative of the function f is given by

$$f'(x) = x^2 - 6x + 5.$$

Note: you are given the *derivative* function! Answer the following questions about f.

(a) Find an interval over which f is increasing.

Solution: We need to solve the inequality $f'(x) > 0$, so we solve the equality $f'(x) = 0$ and use the test interval technique. Factor to get $x^2 - 6x + 5 = (x-1)(x-5) = 0$, so we have three intervals to test, $(-\infty, 1), (1, 5), (5, \infty)$. It follows quickly that $f'(x) > 0$ on the two infinite intervals, $(-\infty, 1)$ and $(5, \infty)$.

(b) Find the location of a relative maximum of f.

Solution: We have the stationary points $x = 1$ and $x = 5$ from part 1 above. We just need to apply the second derivative test. Since $f''(x) = 2x - 6$, it follows that $f''(1) = -4$ and $f''(5) = 4 > 0$. Thus $x = 1$ is the location of a relative maximum.

(c) Find the location of a relative minimum of f.

Solution: From the work done in part 2 above, $x = 5$ is the location of a relative minimum.

(d) Find an interval over which f is concave upwards.

Solution: Here we need to solve the inequality $f''(x) > 0$. This is easy, by the test interval technique or by staring at $2x - 6 > 0$. Thus f is concave upwards on $[3, \infty)$.

(e) Suppose $f(1) = 3$. Find $f(2)$.

Solution: By antidifferentiating, $f(x) = x^3/3 - 3x^2 + 5x + c$ for some constant c. Thus, $f(1) = 1/3 - 3 + 5 + c$ which implies that $c = 2/3$. Then $f(2) = 8/3 - 3 + 5 + 2/3 = 4/3$.

10. (20 points) Suppose the functions f and g are differentiable and their values at certain points are given in the table. The next four problems refer to these functions f and g. Notice that, for example, the entry 1 in the first row and third column means that $f'(0) = 1$. Note also that, for example, if $K(x) = f(x) - g(x)$, then $K'(x) = f'(x) - g'(x)$ and $K'(4) = f'(4) - g'(4) = 5 - 10 = -5$. Answer each of the questions below about functions that can be build using f and g.

x	$f(x)$	$f'(x)$
0	2	1
1	2	3
2	5	4
3	1	2
4	3	5
5	6	4
6	0	5
7	4	1

x	$g(x)$	$g'(x)$
0	5	5
1	7	3
2	4	6
3	2	6
4	6	10
5	3	3
6	1	2
7	0	1

(a) The function h is defined by $h(x) = f(g(x))$. Use the chain rule to find $h'(4)$.

Solution: By the chain rule, $h'(x) = f'(g(x)) \cdot g'(x)$, so $h'(4) = f'(g(4)) \cdot g'(4) = f'(6) \cdot g'(4) = 5 \cdot 10 = 50$.

(b) The function k is defined by $k(x) = f(x) \cdot g(x)$. Use the product rule to find $k'(2)$.

Solution: By the product rule, $k'(x) = f'(x)g(x) + g'(x)f(x)$, so $k'(2) = f'(2)(g(2) \cdot g'(2)f(2) = 4 \cdot 4 + 5 \cdot 6 = 46$.

(c) The function H is defined by $H(x) = g(g(x))$. Use the chain rule to find $H'(2)$.

Solution: By the chain rule, $H'(x) = g'(g(x)) \cdot g'(x)$, so $H'(2) = g'(g(2)) \cdot g'(2) = g'(4) \cdot g(2) = 10 \cdot 6 = 60$.

(d) Let $Q(x) = f(f(x) - g(x))$. Find $Q'(5)$.

Solution: By the chain rule, $Q'(x) = f'(f(x) + g(x)) \cdot (f'(x) + g'(x))$, so $Q'(5) = f'(f(5) + g(5)) \cdot (f'(5) + g'(5)) = f'(6 - 3) \cdot (4 - 3) = f'(3) = 2$.

(e) Find the derivative of the function f/g at the point $x = 4$.

Solution: By the quotient rule, $\frac{d}{dx} f(x)/g(x) = \frac{f'(x)g(x) - g'(x)f(x)}{g(x)^2}$, so the derivative at $x = 4$ is $\frac{f'(4)g(4) - g'(4)f(4)}{g(4)^2} = \frac{5 \cdot 6 - 10 \cdot 3}{36} = 0$.

11. (10 points) Compute each of the following derivatives.

    (a) $\frac{d}{dx}\sqrt{x^2+1}$
    Solution: $\frac{d}{dx}\sqrt{x^2+1} = \frac{1}{2}(x^2+1)^{-1/2}\cdot 2x = x(x^2+1)^{-1/2} = \frac{x}{\sqrt{x^2+1}}$.

    (b) $\frac{d}{dx}\ln(x^4+1)$
    Solution: $\frac{d}{dx}\ln(x^4+1) = \frac{4x^3}{x^4+1}$

12. (20 points) Compute the following antiderivatives.

    (a) $\int 6x^2 - 5x - 1dx$
    Solution: $\int 6x^2 - 5x - 1dx = 2x^3 - 5x^2/2 - x + C$

    (b) $\int 6x^{\frac{5}{2}} + x^{-\frac{1}{2}}dx$
    Solution: $\int 6x^{\frac{5}{2}} + x^{-\frac{1}{2}}dx = 6\cdot 2/7\cdot x^{7/2} + 2x^{1/2} + c$

    (c) $\int \frac{3x^2+2x-1}{x}dx$
    Solution: $\int \frac{3x^2+2x-1}{x}dx = \int 3x + 2 - 1/xdx = 3x^2/2 + 2x - \ln|x| + c$

    (d) $\int \frac{4x+1}{2x^2+x-3}dx$
    Solution: $\int \frac{4x+1}{2x^2+x-3}dx = \ln|x^2+x-3| + c$

13. (20 points) Compute the following integrals.

    (a) $\int_0^2 -3x^2e^{-x^3}\,dx$

    Solution: $\int_0^2 -3x^2e^{-x^3}\,dx = -e^{-x^2}|_0^2 = -e^{-2^3} - (-e^0) = 1 - e^{-8} \approx$ .99966.

    (b) $\int_0^5 (2x+1)\sqrt{x^2+x+5}\,dx$

    Solution: By the method of substitution $\int_0^5 (2x+1)\sqrt{x^2+x+5}\,dx = \frac{2}{3}(x^2+x+5)^{3/2}|_0^5 = \frac{2}{3}(35^{32/} - 5^{2/3} \approx \frac{2}{3}195.88 \approx 130.588.$

14. (10 points) Find the largest interval over which $f(x) = 4x^3 + 39x^2 - 42x$ is decreasing.

    Solution: We need to solve the inequality $f'(x) = 12x^2+78x-42 < 0$. This is equivalent to $2x^2+13x-7 < 0$. But the quadratic factors into $(2x-1)(x+7)$, so the interval where f is decreasing is $(-7, 1/2)$.

15. (10 points) Find a function $G(x)$ whose derivative is $3x^2 - 7x$ and whose value at $x = 4$ is 11.

    Solution: Antidifferentiating gives $G(x) = \int 3x^2 - 7xdx = x^3 - 7x^2/2 + C$, so C must satisfy $G(4) = 4^3 - 7 \cdot 16/2 + C = 11$, which implies that $C = 3$. Thus, $G(x) = x^3 - 7x^2/2 + 3$.

16. (10 points) Find the area of the region bounded by $y = x^3 + 1$, the x-axis, and the lines $x = 0$ and $x = 4$.

    Solution: We need to measure the growth of an antiderivative of y over the interval $[0, 4]$. IE, $\int_0^4 x^3 + 1dx = \frac{1}{4}x^4 + x|_0^4 = \frac{1}{4}4^4 + 4 - (0^4 + 0) = 4^3 + 4 = 68$.

17. (10 points) Find the area A of the region caught between the graphs of the functions

    $$f(x) = -x^2 + 4x \text{ and } g(x) = -3x + 6.$$

    Solution: First we need to solve $f(x) = g(x)$, ie, $-x^2 + 4x = -3x + 6$. This is equivalent to $x^2 - 7x + 6 = 0$ which we can solve by factoring to get $x = 1$ and $x = 6$. Then we integrate the difference function from 1 to 6. Therefore $A = \int_1^6 -x^2 + 7x - 6dx = -x^3/3 + 7x^2/2 - 6x|_1^6 == 6^3/3 + 7 \cdot 6^2/2 - 6 \cdot 6 - (-1/3 + 7/2 - 6) = 18 + 17/6 = 20\frac{5}{6}$

18. (15 points) An apartment complex has 100 two-bedroom units for rent all at the same price. The monthly profit from renting x units is given by

    $$P(x) = -10x^2 + 1760x - 50000$$

    dollars. Find the number of units that should be rented out to maximize the profit. What is the maximum monthly profit realizable?

    Solution: Compute $P'(x)$ to get $P'(x) = -20x+1760$. The only critical point is $x = 1760/20 = 88$, so the complex should rent 88 apartments to maximize the profit. The maximum profit realizable is $P(88) = -10 \cdot 88^2 + 1760 \cdot 88 - 50000 = 27440$.