Math 1120 **Calculus** **Final Exam**

December 11, 2001 Your name ____________________________

The value of each problem is shown. You must show your work on the problems. The total number of points available is 246.

1. (8 points) Consider the function f defined by:

$$f(x) = \begin{cases} 2x^2 - 3 & \text{if } x < 0 \\ 5x^2 - 3x & \text{if } x \geq 0 \end{cases}$$

   Find an equation for the line that is tangent to the graph of f at the point $(3, f(3))$.

   Solution: Since $f'(x) = 10x - 3$ for $x > 0$, $f'(3) = 27$. Also, $f(3) = 45 - 9 = 36$, so the line is $y - 36 = 27(x - 3)$ or $y = 27x - 45$.

2. (8 points) Again referring to the function f defined in problem 1, what is the slope of the line joining the points $(-2, f(-2))$ and $(2, f(2))$?

   Solution: The two points are $(-2, 5)$ and $(2, 14)$, so the slope is $\frac{14-5}{2+2} = 9/4$.

3. (8 points) What is the distance between the point $(6, 8)$ and the midpoint of the segment joining the points $(2, 3)$ and $(10, -7)$?

   Solution: The midpoint of the segment is $(6, -2)$, so the distance is $d = \sqrt{0^2 + 10^2} = 10$.

4. (8 points) Find an equation for each horizontal asymptote of $r(x)$?

$$r(x) = \frac{(x+4)(x^2-1)(3x^2-4)}{(x^2+x-12)(x-1)^4}$$

   Solution: The degree of the denominator is greater than that of the numerator, so the horizontal asymptote is $y = 0$.

5. (8 points) Referring again to the function $r(x)$ in the previous problem, find an equation for each vertical asymptote of $r(x)$?

   Solution: The quadratic in the denominator factors, $x^2+x-12 = (x+4)(x-3)$. The $x + 4$ factor in the denominator cancels with the $x + 4$ factor in the numerator, leaving just the factors $x - 1$ and $x - 3$, so the vertical asymptotes are $x = 1$ and $x = 3$.

6. (8 points) The line tangent to the graph of a function f at the point $(1, 5)$ on the graph also goes through the point $(0, 8)$. What is $f'(1)$?

   Solution: The slope of the line through $(1, 5)$ and $(0, 8)$ is -3.

7. (8 points) What is the slope of the tangent line to the graph of $f(x) = e^{2x}$ at the point $(1, e^2)$?

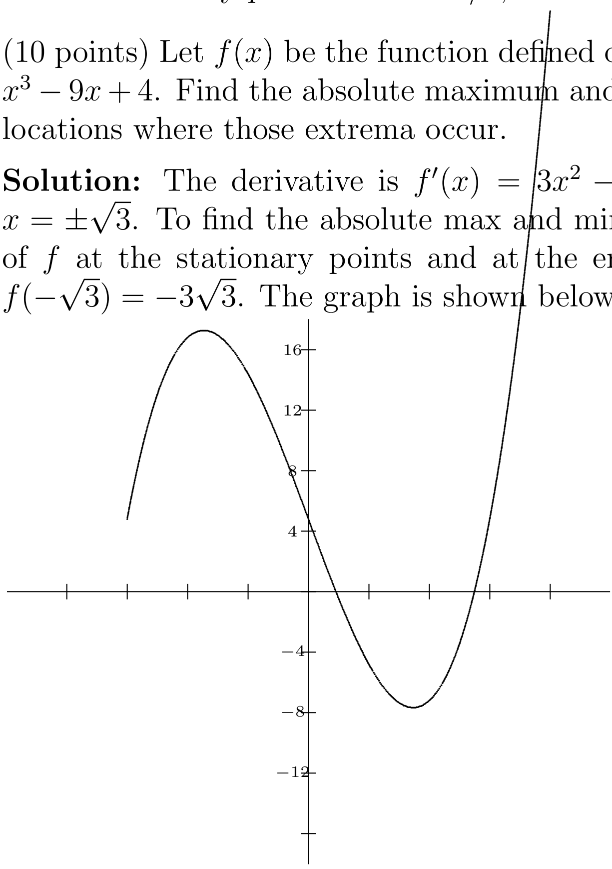
   Solution: The derivative is $f'(x) = 2e^{2x}$ whose value of at $x = 1$ is $f'(1) = 2e^2$.

8. (10 points) Let $g(x) = (2x - 3)^2(x + 1)^2$. Find $g'(x)$ and the critical points of g. Express g' in factored form.

   Solution: Use the product rule to compute g': $g'(x) = 2(2x - 3)^{2-1} \cdot 2(x + 1)^2 + 2(x + 1)^1 \cdot (2x - 3)^2$. Factor out the common terms to get $g'(x) = (2x - 3)(x+1)(4(x+1)+2(2x-3) = (2x-3)(x+1)(8x-2) = 2(2x-3)(x+1)(4x-1)$. So the stationary points are $x = 3/2, x = -1$, and $x = 1/4$.

9. (10 points) Let $f(x)$ be the function defined on $[-3, 4]$ by the equation $f(x) = x^3 - 9x + 4$. Find the absolute maximum and absolute minimum of f and the locations where those extrema occur.

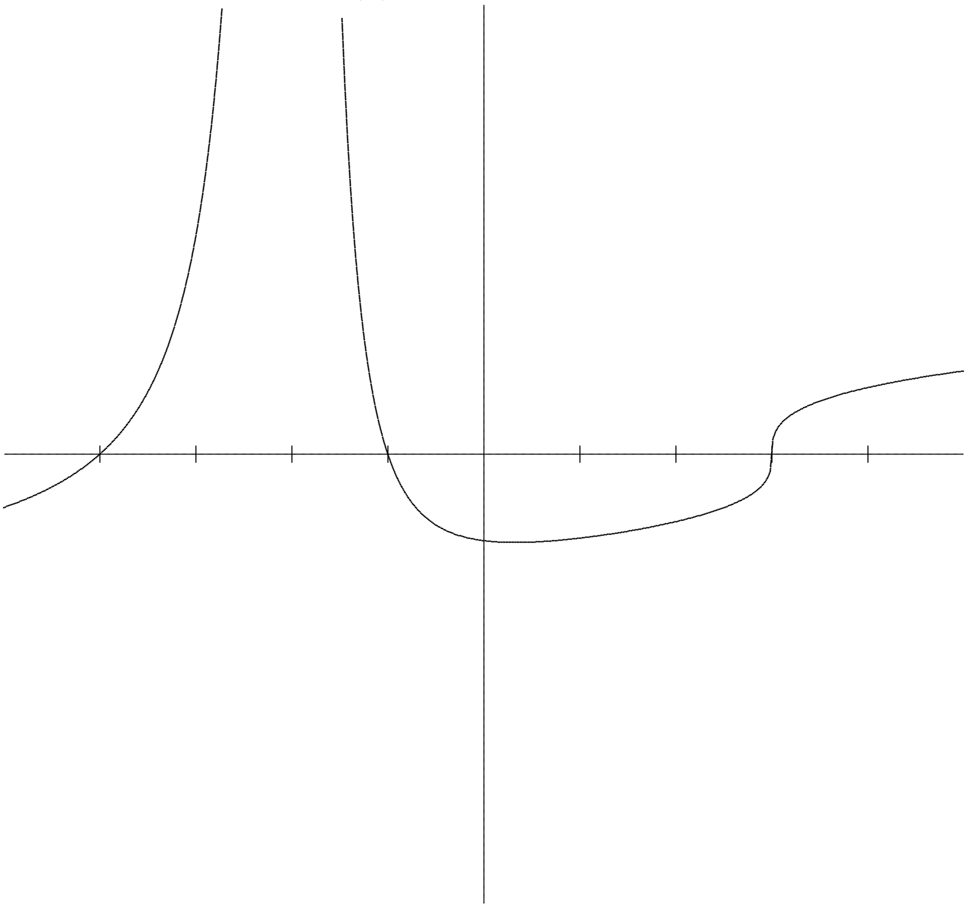
   Solution: The derivative is $f'(x) = 3x^2 - 9$, so the stationary points are $x = \pm\sqrt{3}$. To find the absolute max and min we have to compare the values of f at the stationary points and at the endpoints, -3 and 4. Note that $f(-\sqrt{3}) = -3\sqrt{3}$. The graph is shown below.



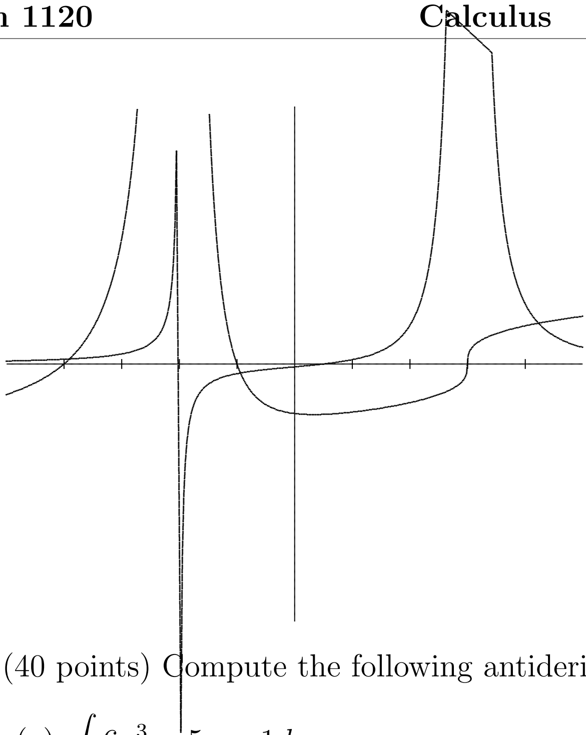
10. (15 points) Suppose $f''(x) = (x-5)(2x+3)(x-3)(2x+9)$. Find the intervals over which f is concave upwards. Note that the second derivative has already been found for you.

    Solution: The four critical points are $x = -9/2, -3/2, 3$, and 5. Use the test interval technique to solve the inequality $f''(x) > 0$. You could use the test points $x = -5, -2, 0, 4$, and 6. You find that $f''(-5), f''(0)$, and $f''(6)$ are positive. Thus $f(x)$ is concave upwards on each of the intervals $(-\infty, -9/2), (-3/2, 3), (5, \infty)$.

11. (10 points) The graph of a function $G(x)$ is shown below. Sketch the graph of the derivative function $G'(x)$ on the same coordinate system.



    Solution: The function $G(x)$ has a horizontal tangent line at roughly $x = 1/2$, so the function $G'(x)$ must have a zero there. Since the tangent line at $x = 3$ is vertical, $G'(x)$ has a vertical asymptote at 3. Below are sketches of both $G(x)$ and $G'(x)$ on the same set of axes.



12. (40 points) Compute the following antiderivatives.

(a) $\int 6x^3 - 5x - 1dx$

Solution: use the power rule to antidifferentiate: $\int 6x^3 - 5x - 1dx = 3x^3/2 + 5x^2/2 - x + C$.

(b) $\int 6x^{\frac{3}{2}} + x^{-\frac{1}{2}}dx$

Solution: Again use the power rule to antidifferentiate: $\int 6x^{\frac{3}{2}}+x^{-\frac{1}{2}}dx = 6 \cdot 2/5 \cdot x^{5/2} + 2x^{1/2} + C$.

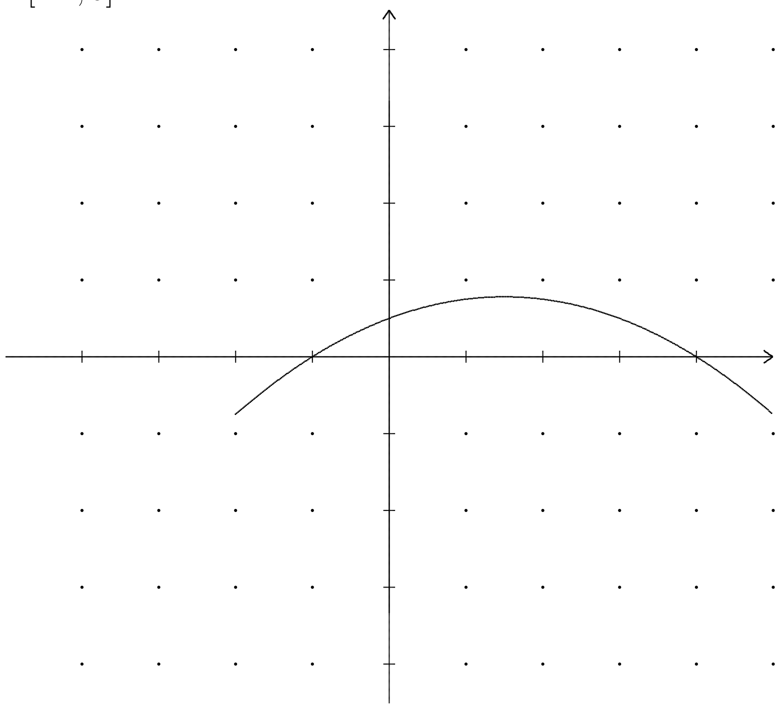
(c) $\int \frac{3x^3 + 2x - 1}{x}dx$

Solution: Simplify by dividing first and then antidifferentiate term by term to get $\int \frac{3x^3 + 2x - 1}{x}dx = \int 3x^2 + 2 - 1/xdx = x^3+2x-\ln|x|+C$.

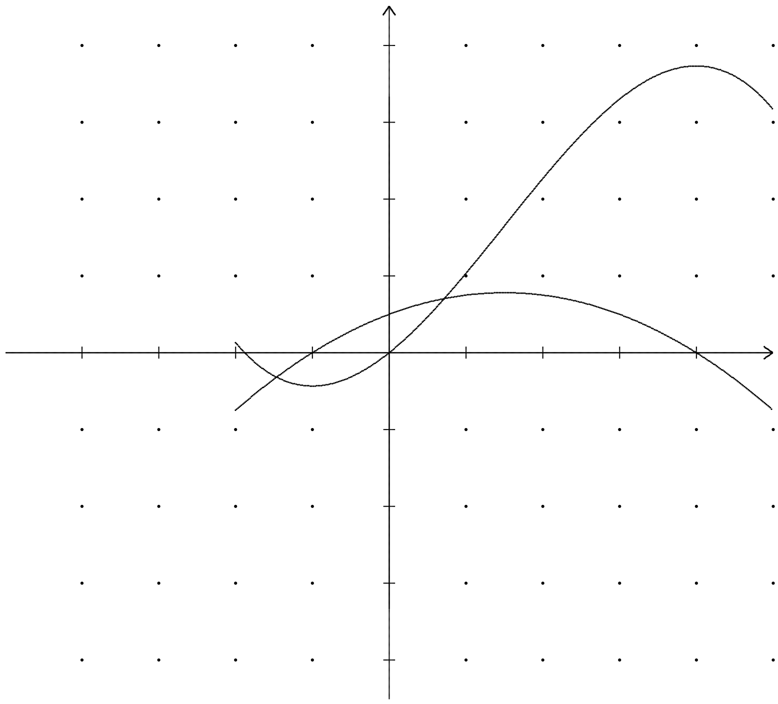
(d) $\int \frac{2x + 1}{x^2 + x - 3}dx$

Solution: $= \ln|x^2 + x - 3| + C$.

13. (10 points) The graph of $f'(x)$ is given below. Suppose $f(0) = 0$. Sketch the graph of $f(x)$ on the same coordinate system. Notice that f' is defined only on the interval $[-2, 5]$.



Solution: Notice that the graph of f' seems to be a parabola that open upwards and has two zeros, at $x = -1$ and $x = 4$. We might therefore guess that f' has a symbolic representation that is close to $-(x+1)(x-4)$. This doesn't quite do it but it gets us in the ballpark. It shows that we should try f as a cubic polynomial. Antidifferentiating $-(x+1)(x-4)$ results in $f(x) = -x^3/3 + 3x^2/2 - 4x$ which is increasing precisely where f' is positive, decreasing exactly where f' is negative and satisfies $f(0) = 0$. The only problem is f' is much flatter than $-(x+1)(x-4)$, so f must be flatter as well. The point is that the behavior of f' near -1 shows that f must have a relative minimum at $x = -1$ and the behavior of f' near 4 implies that f has a relative maximum at 4. The graphs of both f' and f are shown below.



14. (10 points) Find a function $G(x)$ that satisfies $G'(x) = 3x^2 - 7x$ and $G(4) = 9$.

    Solution: Antidifferentiating G' gives $G(x) = x^3 - 7x^2/2 + C$ where C is a constant that has to be determined. Since $G(4) = 9 = 4^3 - 7 \cdot 4^2/2 + C$, it follows that $C = 9 - 8 = 1$, and that $G(x) = x^3 - 7x^2/2 + 1$.

15. (40 points) Compute each of the following derivatives.

    (a) $\frac{d}{dx}\sqrt{x^4+1}$
    Solution: By the chain rule, $\frac{d}{dx}\sqrt{x^4+1} = 4x^3 \cdot \frac{1}{2}(x^4+1)^{-1/2} = 2x^3/\sqrt{x^4+1}$.

    (b) $\frac{d}{dx}[\ln(2x+1)]^3$
    Solution: Again by the chain rule, $\frac{d}{dx}[\ln(2x+1)]^3 = 3 \cdot \ln(2x+1) \cdot \frac{2}{2x+1} = \frac{6(\ln(2x+1))^2}{2x+1}$.

    (c) $\frac{d}{dx}x^2e^{2x+1}$
    Solution: By the product rule, $\frac{d}{dx}x^2e^{2x+1} = 2xe^{2x+1} + 2x^2e^{2x+1} = 2xe^{2x+1}(1+x)$.

(d) $\dfrac{d}{dx}\dfrac{2x^2-3}{2x+3}$

Solution: By the quotient rule, $\dfrac{d}{dx}\dfrac{2x^2-3}{2x+3} = \dfrac{4x(2x+3)-2(2x^2-3)}{(2x+3)^2} = 2\dfrac{2x^2+6x+3}{(2x+3)^2}$, the numerator of which does not factor.

16. (20 points) Compute the following integrals.

(a) $\displaystyle\int_0^2 2xe^{-x^2}dx$

Solution: $\displaystyle\int_0^2 2xe^{-x^2}dx = -e^{-x^2}|_0^2 = -e^{-4}-(-1) = 1-1/e^4 \approx .9816.$

(b) $\displaystyle\int_0^5 (2x-1)\sqrt{x^2-x+5}\,dx$

Solution: By substitution, let $u = x^2-x+5$. Then $du = 2x-1dx$, and we have $\displaystyle\int_0^5 (2x-1)\sqrt{x^2-x+5}\,dx = \int \sqrt{u}\,du = \frac{2}{3}(u^{3/2} = \frac{2}{3}(x^2-x+5)^{3/2}|_0^5 = \frac{2}{3}(125-5\sqrt{5}) \approx 75.88.$

17. (10 points) Find the area of the region bounded by $y = x^{3/2}$, the x-axis, and the lines $x = 0$ and $x = 4$.

Solution: The integral is $\displaystyle\int_0^4 x^{3/2}dx = \frac{2}{5}x^{5/2}|_0^4 = \frac{64}{5} = 12.8.$

18. (15 points) Find the area A of the region caught between the graphs of the functions

$$f(x) = -x^2+4x \text{ and } g(x) = -2x+5.$$

Solution: First find the two points of intersection of the graphs by solving $-x^2+4x = -2x+5$ to get $x = 1$ and $x = 5$. Then integrate the difference $f(x)-g(x)$ from $x = 1$ to $x = 5$. Thus $A = \displaystyle\int_1^5 -x^2+4x+2x-5dx = -x^3/3+3x^2-5x|_1^5 = 10\frac{2}{3}.$