December 14, 2015 **Name** ______________________________

The problems count as marked. The total number of points available is 280. Throughout this test, **show your work.** Using a calculator or a technique like L'hospital's Rule to circumvent ideas discussed in class will generally result in no credit. Throughout the test, express each measurement using the correct units.

1. (10 points) The line tangent to the graph of a function f at the point $(2, 9)$ has slope -2. Find an equation for the tangent line in slope-intercept form.

   Solution: Since $y - 9 = -2(x - 2)$, we have $y = -2x + 13$.

2. (10 points) Find an equation in slope-intercept form for the line tangent to the graph of $f(x) = x^{-2} - 3x$ at the point $(2, f(2))$?

   Solution: The derivative is $f'(x) = -2x^{-3} - 3$ whose value of at $x = 2$ is $f'(2) = -1/4 - 3 = -3.25$. Thus the line is $y - (-5.75) = -3.25(x - 2)$, which, in slope intercept form is $y = -3.25x + 0.75$ or $y = -13x/4 + 3/4$.

3. (10 points) Suppose f is a linear function. In other words, it can be written in the form $f(x) = mx + b$ for some constants m and b. Suppose also that $f(8) - f(2) = 12$. What is m?

   Solution: Since m represents the slope of the line, we have $m = \frac{f8) - f(2)}{8 - 2} = \frac{12}{6} = 2$.

4. (10 points) Find all values of x for which $|x - 1| + |x - 5| + |x - 11| = 12$.

   Solution: Let $g(x) = |x - 1| + |x - 5| + |x - 11| = 12$. Trying a few numbers in the range $[1, 11]$, we get $g(2) = |1| + |-3| + |-9| = 13$, $g(3) = |2| + |-2| + |-8| = 12$, so we've found one solution. Playing a bit more we can see that for any x in the range $[1, 11]$, the value of $|x - 1| + |x - 11|$ is $11 - x + x - 1 = 10$. So we need to make $|x - 5|$ have the value 2. Both $x = 3$ and $x = 7$ do this.

5. (10 points) Find an equation for the line tangent to the graph of $f(x) = \ln(3x+1)$ at the point (0, 0)?

   Solution: The derivative is $f'(x) = \frac{3}{3x+1}$ whose value of at $x = 0$ is $f'(0) = 3$. Thus the line is $y - 0 = 3(x - 0)$, which, in slope intercept form is $y = 3x$.

6. (20 points) The marketing department of a large company has decided that the demand function for the new super-smartphones they manufacture is given by $p = 12 - x^2$ where x is the number of (in thousands) produced each week and p is the price per phone in hundreds of dollars.

   (a) Build the weekly revenue function $R(x)$ for the phone.

   Solution: $R(x)$ is the product of x and $p(x)$, so $R(x) = x \cdot p = x(12-x) = 12x - x^3$.

   (b) Find the marginal revenue.

   Solution: $R'(x) = 12 - 3x^2 = 3(4 - x^2)$.

   (c) How should the company price its phones to maximize the revenue?

   Solution: At $x = 2$, $R'(x) = 0$, and the revenue is $2(12 - 2^2) = 16$. The company's revenue is 1.6 million dollars. The price p that maximizes R is $12 - 2^2 = 8 = \$800$ per phone.

   (d) Marketing has also established that the cost associated with the production of x phones is $C(x) = 2.95x + 250$ again measured in hundreds of dollars. What is the average cost function?

   Solution: $\overline{C}(x) = C(x)/x = 2.95 + 250/x$.

   (e) What is the marginal average cost?

   Solution: By the quotient rule, $\overline{C'}(x) = \frac{d}{dx}\ C(x)/x = \frac{C'(x)x - C(x)}{x^2} = -250/x^2$.

7. (40 points) Consider the function f below.

$$f(x) = \begin{cases} 1 - x & \text{if } x < -2 \\ x^2 - 1 & \text{if } -2 \leq x < 2 \\ 5 - x & \text{if } 2 < x \end{cases}$$

(a) What is the domain of f?

Solution: The function is defined for all values of x except 2, so the domain is $D = (-\infty, 2) \cup (2, \infty)$.

(b) Discuss the continuity of f at the point $x = -2$.

Solution: f is continuous at -2 because the left and right hand limits are both 3 and $f(-2) = 3$ as well.

(c) Build the sign chart for f.

Solution: The zeros of f are $x = \pm 1$ and $x = 5$. Since f is undefined at 2 we have to include 2 in the sign chart. So f is positive on $(-\infty, -1) \cup (1, 2) \cup (2, 5)$, and negative on $(-1, 1) \cup (5, \infty)$.

(d) Find the derivative of f. Of course it will have to be piecewise defined, just like f is.

Solution:

$$f'(x) = \begin{cases} -1 & \text{if } x < -2 \\ 2x & \text{if } -2 < x < 2 \\ -1 & \text{if } 2 < x \end{cases}$$

(e) Find the intervals over which the function f is increasing.

Solution: Note that f is increasing on $(0, 2)$, precisely where $f'(x) > 0$.

8. (30 points) Let R denote the region defined by

$$R = \{(x, y) |\ 1 \leq x \leq e \text{ and } 0 \leq y \leq 1/x\}.$$

In other words, R is the region bounded by the lines $x = 1, x = e$ and $y = 0$ (ie, the x axis), and the function $y = 1/x$.

(a) Estimate the area of R by finding the area of a rectangular region inside R. The area of this rectangle is less than the area of R.

Solution: One such rectangle has width $e - 1$ and height $1/e$, so area $1 - 1/e \approx 0.632$.

(b) Estimate the area of R by finding the area of a rectangular region that encloses R. The area of this rectangle is greater than the area of R.

Solution: One such rectangle has width $e - 1$ and height 1, so area $e - 1 \approx 1.718$.

(c) Compute the area of R.

Solution: An antiderivative of $1/x$ is $\ln(x)$, which grows by $\ln(e) - \ln(1) = 1 - 0 = 1$ over the interval in question.

(d) How does the area of R change is we replace $x = e$ with $x = e^3$ in the definition of R?

Solution: The growth of $\ln(x)$ over $[1, e^3]$ is 3.

9. (50 points)

   (a) Find $\int_2^3 (x-2)(x^2+2x+4)\ dx$

   Solution: Multiply this out to get $\int_2^3 x^3 - 8\ dx$, which is $x^4/4 - 8x|_2^3 = 8.25$.

   (b) Find $\int (x-2)^{10}(x+1)\ dx$

   Solution: Use the substitution $u = x - 2$ to get $\int u^{10}(u+3)\ du = \int u^{11} + 3u^{10}\ du$. We get $\frac{(x-2)^{12}}{12} + \frac{3(x-2)^{11}}{11} + C$.

   (c) Find $\int (x-2)^2\ dx$

   Solution: $\int (x-2)^2\ dx = \int x^2 - 4x + 4 = x^3/3 - 2x^2 + 4x + C$. Alternatively, we could use substitution to get $(x-2)^3/3 + C$.

   (d) Find $\int_{-2}^2 \sqrt{4-x^2}\ dx$

   Solution: Draw the graph of the function to realize that the region R bounded by the function is just a semicircle of radius 2, so the integral is $\frac{1}{2}\pi \cdot 2^2 = 2\pi$.

   (e) Find $\int x^3 - x^{-2} + x^{-1}\ dx$

   Solution: $\int x^3 - x^{-2} + x^{-1}\ dx = x^4/4 + x^{-1} + \ln(x) + C$.

   (f) Evaluate $\int_1^3 \frac{x^3-2x^2+x}{x}\ dx$

   Solution: Its just $\int_1^3 x^2 - 2x + 1\ dx = x^3/3 - x^2 + x|_1^3 = (9 - 9 + 3) - (1/3 - 1 + 1) = 8/3$.

   (g) Evaluate $\int_0^8 \frac{d(x-5)^2}{dx}\ dx$

   Solution: Its just $(x-5)^2|_0^8 = 3^2 - (-5)^2 = 9 - 25 = -16$.

   (h) Evaluate $\int_0^4 \frac{3x^2}{x^3+5}\ dx$

   Solution: Notice that $\frac{d}{dx}x^3 + 5 = 3x^2$, so $\int_0^4 \frac{3x^2}{x^3+5}\ dx = \ln(x^3+5)|_0^4 = \ln(69) - \ln 5 \approx 2.625$.

10. (10 points) Let $h(x) = \frac{\sqrt{(x-4)(x-2)(2x+9)}}{x^2-100}$. Write the domain of h in interval notation.

    Solution: There are two things to guard against, division by zero and having a negative sign in the radical. Division by zero happens when either $x = 10$ or $x = -10$. So we need the sign chart for $(x-4)(x-2)(2x+9)$, which clearly has zeros at $x = 2, x = 4$, and $x = -7/2$. The sign chart reveals that the product is positive over $(-9/2, 2)$ and $(4, \infty)$. Therefore, the answer we seek is $[-9/2, 2] \cup [4, 10) \cup (10, \infty)$.

11. (20 points) Consider the function $F(x) = (x^2+1)^{\frac{3}{2}}$.

    (a) Find $F'(x)$.

    Solution: By the chain rule, $F'(x) = \frac{3}{2}(x^2+1)^{\frac{1}{2}} \cdot 2x$.

    (b) Use your work in (a) to compute $\int \sqrt{x^2+1} \cdot 2x\ dx$.

    Solution: Of course, it is just $\frac{2}{3}(x^2+1)^{\frac{3}{2}} + C$.

    (c) Find $\int_{-1}^{1} \sqrt{x^2+1} \cdot 2x\ dx$.

    Solution: $\frac{2}{3}(x^2+1)^{\frac{3}{2}}|_{-1}^{1} = 0$.

    (d) Explain why you can get such an unusual answer as you did in part (c).

    Solution: The function $\sqrt{x^2+1} \cdot 2x$ is negative from -1 to 0, so the integral and the area are not the same. In fact it is an example of what is called an *odd* function. Its graph is symmetric with respect to the origin. So the part where its positive is exactly cancelled out by the part where its negative.

12. (15 points) It takes 14 years for a certain \$2000 continuously-compound investment to triple in value.

    (a) How long does it take before its value is \$4000?

    Solution: The rate of interest is $r = \ln(3)/14 = 0.07847$. It takes such an investment $\ln(2)/r = 8.833$ years to double.

    (b) How long does it take to reach a value of \$8000?

    Solution: If it doubles in 8.833 years, it quadruples in 17.66 years.

13. (15 points) It takes 7 years for a 60 units sample of *Reiterarium* to lose two-thirds of its radioactivity.

    (a) What is the half-life of Reiterarium?

    Solution: We have $Q(t) = Q_0 e^{-kt}$. Replacing each parameter with what is known, we have $20 = 60e^{-7k}$, which we can solve for k. Taking logs of both sides yields $\ln(e^{-7k}) = -7k = \ln(1/3) = \ln(1) - \ln(3) = -\ln(3)$, so $k = \ln(3)/7 \approx 0.15694$. So the half-life is $t = \ln(2)/k \approx 4.41$ years.

    (b) How long does it take before the 60 units is reduced to 2 units?

    Solution: It takes $t = -\ln(30)/-k \approx 21.67$ years.

14. (25 points) Let $f(x) = e^{8-2x^2}$.

    (a) Compute $f'(x)$
    Solution: $f'(x) = -4xe^{8-2x^2}$.

    (b) Compute $f''(x)$
    Solution: By the product rule, $f''(x) = -4e^{8-2x^2} + 16x^2e^{8-2x^2}$.

    (c) What is the slope of f's steepest positive-sloped tangent line?
    Solution: You are being asked to maximize $f'(x)$. So you find its critical points, which are easy to find, $x = -1/2$ and $x = 1/2$. Building the sign chart for $f''(x)$, we can see that f' has a relative maximum at $x = -1/2$.

15. (15 points) Let $f(x) = \ln[(x^2+1)(x^3+x^2)(\frac{1}{x+2})]$.

    (a) Find the derivative of $f(x)$.
    Solution: First, rewrite f as $f(x) = \ln(x^2+1) + \ln((x^2)(x+1)) + \ln(x+2)^{-1} = \ln(x^2+1) + \ln(x^2) + \ln(x+1) + \ln(x+2)^{-1}$. At this point the differentiation is much easier: $f'(x) = \frac{2x}{x^2+1} + \frac{2}{x} + \frac{1}{x+1} - \frac{1}{x+2}$.

    (b) What is $f'(1)$?
    Solution: $f'(1) = 1 + 2 + 1/2 - 1/3 = 19/6$

    (c) Find an equation for the line tangent to f at the point $(1, f(1))$. Leave your answer in terms if the ln function.
    Solution: Since $f(1) = \ln(2 \cdot 2 \cdot 1/3) = \ln(4/3) = \ln(4) - \ln(3)$, we have $y - \ln(4/3) = 19(x-1)/6$ and the slope-intercept form is easy to get.