December 14, 2016 **Name** ____________________

The problems count as marked. The total number of points available is xxx. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (48 points) Find the following antiderivatives.

   (a) $\int 2x - 5\ dx$

   Solution: $x^2 - 5x + C$.

   (b) $\int 9x^2 - 4x - 2/x\ dx$

   Solution: $3 \cdot x^3 - 2 \cdot x^2 - 2 \ln x + C$.

   (c) $\int \dfrac{3x^3 + 2x^2 - x}{x}\ dx$

   Solution: $\int (3x^3 + 2x^2 - x)/x\ dx = \int 3x^2 + 2x - 1 dx = x^3 + x^2 - x + C$.

   (d) $\int \dfrac{2x+3}{x^2 + 3x - 3}\ dx$

   Solution: By substitution, $(u = x^2 + 3x - 3)$, $\int \frac{2x+3}{x^2+3x-3}\ dx = \ln |x^2 + 3x - 3| + C$.

   (e) $\int 6x^5 (x^6 + 3)^7\ dx$

   Solution: By substitution with $u = x^6 + 3$, $\int 6x^4 (x^6 + 3)^7\ dx = \frac{(x^6+3)^8}{8} + C$.

   (f) $\int x^2 e^{x^3}\ dx$

   Solution: By substitution with $u = x^3, du = 3x^2$, $\int e^u\ du = e^u + C = (1/3)e^{x^3} + C$.

   (g) $\int \dfrac{2x+3}{x-1}\ dx$. Try substitution with $u = x - 1$.

   Solution: By substitution with $u = x - 1, du = dx$, $\int \frac{2u+5}{u}\ du = \int 2 + \frac{5}{u}\ du = 2u + 5 \ln u = 2((x-1) + 5 \ln(x-1) + C$.

   (h) $\int (x+2)^4 (x-2)\ dx$

   Solution: By substitution with $u = x + 2$, $\int (x+2)^4 (x-2)\ dx = \int u^4 (u - 4)\ du = \int u^5 - 4u^4\ du = u^6/6 - 4u^5/5 + C = (x+2)^6/6 - 4(x+2)^5/5 + C$.

2. (10 points) Suppose you're given a function f to differentiate. You do so and you get $f'(x) = (x+3)(x+1)(x^2)(x-4)$. You conclude that f has four critical points, $x = -3, -1, 0$ and 4. Classify each of these as a) the location of a relative maximum, b) the location of a relative minimum, or c) an imposter.

   Solution: Because there is no sign change at $x = 0$, we know 0 is an imposter. Since $f'(x)$ changes from negative to positive at 4, we know that 4 is the location of a minimum.

3. (10 points) Consider the function $g(x) = \ln((x-3)(x^2+5x+4))$. Notice that $g(0) = \ln(-3 \cdot 4)$ is not defined because x must be positive in order to take ln of it. Find the domain of $g(x)$ and put your answer in interval notation.

   Solution: Build the sign chart for $(x-3)(x^2+5x+4) = (x-3)(x+4)(x+1)$ to find the domain.

4. (12 points) Let $H(x) = 2x - \ln(4x^2 + 12x + 10)$. Find all the critical points.

   Solution: We need to solve the equation $\frac{8x+12}{4x^2+12x+10} = 2$. This is equivalent to $8x^2 + 16x + 8 = 0$ which has repeated roots, $x = -1$.

5. (20 points) Consider the function $f(x) = (2x-4)e^{x^2}$.

   (a) Use the product rule to find $f'(x)$.
   Solution: $f'(x) = 2e^{x^2} + 2x(2x-4)e^{x^2}$

   (b) List the critical points of f.
   Solution: Factor the expression above to get $(4x^2 - 8x + 2)e^{x^2} = 2e^{x^2}(2x^2 - 4x + 1)$, which has value 0 when $x = 1 - \sqrt{2}/2 \approx 0.293, x = 1 + \sqrt{2}/2 \approx 1.707$. Call the first value α and the second β.

   (c) Construct the sign chart for $f'(x)$.
   Solution: f' is positive on $(-\infty, \alpha)$ and on (β, ∞).

   (d) Write in interval notation the interval(s) over which f is increasing.
   Solution: f is increasing on $(-\infty, \alpha)$ and on (β, ∞).

6. (36 points) Demonstrate your understanding of the product, quotient and chain rules by differentiating each of the given functions. Find the critical points for each function and the intervals over which the function is increasing. You must show your work.

   (a) Let $F(x) = (2x+8)(4x-6)$

   Solution: Note that $F'(x) = 2(4x-6) + 4(2x+8) = 16x + 20$, so the only critical point is $x = -20/16$. Since $f'(x) > 0$ on $(-20/16, \infty)$, we conclude that F is increasing on that interval.

   (b) $G(x) = \dfrac{x^2 - 3x + 15/2}{2x-1}$

   Solution: By the quotient rule, $G'(x) = \frac{(2x-3)(2x-1)-2(x^2-3x+15/2)}{(2x-1)^2} = \frac{2x^2-2x-12)}{(2x-1)^2}$. So the critical points are $x = -2$ and $x = 3$. We can see that $G' > 0$ on both $(-\infty, -2)$ and on $(3, \infty)$. So G is increasing on these two intervals.

   (c) $K(x) = (x^2-4)^{18}$

   Solution: By the chain rule, $K'(x) = 18(x^2-4)^{17} \cdot 2x$, so the critical points are $x = -2, 0, 2$. Build the sign chart for K' to see that K' is positive on both $(-2, 0)$ and $(2, \infty)$. Therefore K is increasing on those two intervals.

7. (10 points) The line tangent to the graph of $g(x)$ at the point $(4, 6)$ has a y-intercept of 9. What is $g'(4)$?

   Solution: The line has slope $(9-6)/(0-4) = -3/4$.

8. (10 points) Find all the points (x, y) on the graph of $h(x) = 2x^2 - 4x$ where the tangent line has a slope equal to 5.

   Solution: Since $h'(x) = 4x - 4$, we can solve $4x - 4 = 5$ for x, which yields $x = 9/4$ and $y = 2 \cdot 81/16 - 36/4 = 18/16 = 9/8$.

9. (15 points) Find a number b such that $\int_b^{2b} x^3\,dx = 60$.

   You must show your work to get any credit on this problem. Guessing b is not enough.

   Solution: $x^4/4|_b^{2b} = \frac{1}{4}(16b^4 - b^4) = \frac{15}{4}b^4 = 60$ and it follows that $b = 2$.

10. (15 points) The region R is bounded by the vertical lines $x = 1$ and $x = e^2$ and by the graph of $f(x) = 1 + \frac{1}{x}$ and the x-axis. Find the area of R.

    Solution: Antidifferentiate to get $x + \ln(x)|_1^{e^2} = 2 + e^2 - 1 = 1 + e^2$.

11. (30 points) Let $A = (6, 6)$, $B = (1, 4)$, $C = (1, 0)$ and $D = (6, 0)$ be the vertices of a quadrilateral in the plane.

    (a) Sketch the figure and use geometry to find the area of $ABCD$

    Solution: The area is 25.

    (b) Find an equation for the linear function (the line) that goes through the points A and B. Give this function the name f.

    Solution: Since we know two points on the line, it follows that $f(x)-4 = \frac{6-4}{6-1}(x-1)$, which we can write as $f(x) = 2x/5 + 18/5$.

    (c) Use calculus to find the area of the region R defined as follows:

    $$R = \{(x, y) : 1 \leq x \leq 6,\ 0 \leq y \leq f(x)\}$$

    Solution: We need to find $\int_1^6 2x/5+18/5\,dx$. This is just $\left.\frac{x^2+18x}{5}\right|_1^6 = 25$.