May 7, 2004 **Name** ____________________

The first six problems count 7 points each (total 42 points) and rest count as marked. There are 207 points available. Good luck.

1. Consider the function f defined by:

$$f(x) = \begin{cases} x^2 - 3 & \text{if } x < 0 \\ 5x - 3 & \text{if } x \geq 0 \end{cases}$$

Find the slope of the line which goes through the points $(-2, f(-2))$ and $(3, f(3))$.

(A) 7/5 **(B)** 11/5 **(C)** 17/5 **(D)** 5 **(E)** 7

Solution: B. The two points on the graph are $(-2, 1)$ and $(3, 12)$ and the slope of the line joining them is $m = 11/5$.

2. The distance between the point $(6.5, 8.5)$ and the midpoint of the segment joining the points $(1, 5)$ and $(2, 7)$ is

(A) $\sqrt{22}$ **(B)** $\sqrt{23}$ **(C)** $5\sqrt{5}/2$ **(D)** $\sqrt{26}$ **(E)** 6

Solution: C. The midpoint of the segment is $(1.5, 6)$, so the distance is $d = \sqrt{5^2 + 2.5^2} = 5\sqrt{5}/2$.

3. Let $f(x) = 2x + 3$ and $g(x) = 3x - 6$. Which of the following does not belong to the domain of $f \circ g$?

(A) 1 **(B)** 2 **(C)** 3 **(D)** 4

(E) The domain of $f \circ g$ is the set of all real numbers.

Solution: E. This composition is defined for every real number.

4. The line tangent to the graph of a function f at the point $(2, 5)$ on the graph also goes through the point $(0, 7)$. What is $f'(2)$?

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Solution: B. The slope of the line through $(2, 5)$ and $(0, 7)$ is -1.

5. What is the slope of the tangent line to the graph of $f(x) = x^{-1}$ at the point (3,1/3)?

   (A) -1 **(B)** $-1/2$ **(C)** $-1/3$ **(D)** $-1/9$ **(E)** $1/3$

   Solution: D. The derivative is $f'(x) = -x^{-2}$ whose value of at $x = 3$ is $f'(2) = -1/9$.

6. The line tangent to the graph of the function $f(x)$ at the point $(2, 5)$ is $2y-3x = 4$. What is $f'(2)$?

   (A) 0 **(B)** $2/3$ **(C)** $3/2$ **(D)** $-2/3$ **(E)** $-3/2$

   Solution: C. The line has slope 3/2.

7. (15 points) Let $f(x) = \sqrt{2x-1}$.

   (a) Construct $\frac{f(5+h)-f(5)}{h}$

   Solution: $\frac{f(5+h)-f(5)}{h} = \frac{\sqrt{2(5+h)-1}-\sqrt{2(5)-1}}{h}$.

   (b) Simplify and take the limit of the expression in (a) as h approaches 0 to find $f'(5)$.

   Solution: $\lim_{h\to 0} \frac{f(5+h)-f(5)}{h} = \lim_{h\to 0} -\frac{\sqrt{2(5+h)-1}-\sqrt{2(5)-1}}{h}$. Rationalize the numerator to get $\lim_{h\to 0} -\frac{\sqrt{2(5+h)-1}-3}{h} = \lim_{h\to 0} \frac{\sqrt{9+2h}-3}{h} = 1/3$.

   (c) Use the information found in (b) to find an equation for the line tangent to the graph of f at the point $(5, 3)$.

   Solution: $y - 3 = (1/3)(x - 5)$.

8. (20 points) Find the interval(s) where $f(x) = (x-4)(x^2-1)(x+3)$ is positive.

   Solution: The branch points are $x = 4, 1, -1$, and -3. I picked test points $-4, -2, 0$, and 5, and found that $f(-4) > 0$, $f(-2) < 0$, $f(0) > 0$, $f(2) < 0$, and $f(5) > 0$. Therefore, the function f is positive on $(-\infty, -3), (-1, 1)$, and $(4, \infty)$.

9. (15 points) Let $f(x) = 4x^3 + 6x^2 - 24x + 1$.

   (a) Find the interval(s) where f is decreasing.
   Solution: $f'(x) = 12x^2 + 12x - 24$, which has two zeros, $x = -2,\ x = 1$. So, by the test interval technique f' is positive over the intervals $(-\infty, -2)$ and $(1, \infty)$. Thus, f is decreasing over the interval $(-2, 1)$.

   (b) Find all inflection points of f.
   Solution: There is one inflection point, $(-1/2, f(-1/2)) = (-1/2, 14)$

10. (20 points) A ball is thrown upwards from the top of a building that is 200 feet tall. The position of the ball at time t is given by $s(t) = -16t^2 + 36t + 200$, where $s(t)$ is measured in feet and t is measured in seconds.

    (a) What is the position of the ball at time $t = 0$?
    Solution: $s(0) = 200$ feet.

    (b) What is the velocity of the ball at time $t = 0$?
    Solution: $s'(t) = -32t + 36$ and $s'(0) = 36$.

    (c) What is the acceleration of the ball at time $t = 0$?
    Solution: $a(t) = s''(t) = -32$, so $a(0) = -32$.

    (d) What is the velocity of the ball at time $t = 1$?
    Solution: $s'(1) = -32 \cdot 1 + 36 = 4$.

    (e) How many seconds elapse before the ball hits the ground?
    Solution: Solve $-16t^2 + 36t + 200 = 0$ to get $t \approx 4.835$.

    (f) What is the speed of the ball when it hits the ground?
    Solution: $s'(4.83) \approx -118.726$.

    (g) What is the acceleration of the ball at the time it hits the ground?
    Solution: $a(t) = v'(t) = s''(t) = -32\text{ft/sec}^2$.

11. (15 points) Find an equation for the line tangent to the graph of $f(x) = x\ln(x) - x$ at the point $(1, f(1))$.

    Solution: $f'(x) = \ln(x) + x \cdot \frac{1}{x} - 1$ so $f'(1) = 0 + 1 \cdot 1 - 1 = 0$, and since $f(1) = -1$, it follows that the tangent line has the equation $y = -1$.

12. (20 points) Find the absolute maximum value of the function

$$f(x) = x^3 - 6x^2 + 9x - 5$$

    over the interval $[0, 4]$.

    Solution: $f'(x) = 3x^2 - 12x + 9$, which is zero when $3(x-3)(x-1) = 0$, so the critical points are $x = 1$ and $x = 3$. Comparing the values of f at these and the endpoints, we get $f(0) = -5$; $f(1) = -1$; $f(3) = -5$; and $f(4) = -1$. Thus the absolute max is -1 and it occurs twice, at 1 and at 4.

13. (60 points) Find the following antiderivatives and definite integrals.

    (a) $\int 6x^3 - 5x - 1dx$

    Solution: $\int 6x^3 - 5x - 1dx = 3x^4/2 - 5x^2/2 - x + C$.

    (b) $\int 4x^{\frac{5}{2}} + x^{-\frac{1}{2}}dx$

    Solution: $\int 4x^{\frac{5}{2}} + x^{-\frac{1}{2}}dx = 8x^{7/2}/7 + 2x^{1/2} + C$.

    (c) $\int \frac{3x^4 + 2x^2 - 1}{x^2}dx$

    Solution: $\int \frac{3x^4 + 2x^2 - 1}{x^2}dx = \int 3x^2 + 2 - x^{-2}dx = x^3 + 2x + 1/x + C$
    .

    (d) $\int \frac{2x + 1}{x^2 + x - 3}dx$

    Solution: Note that the numerator is the derivative of the denominator. Thus the antiderivative is $\ln|x^2 + x - 3| + C$.

    (e) $\int 5x^4(x^5 + 2)^3dx$

    Solution: By substitution, $\int 5x^4(x^5+2)^3dx = \frac{u^4}{4} + C = (x^5+2)^4/4 + C$, where $u = x^5 + 2$.

(f) $\int_0^1 2x^2 - 3xdx$

Solution: $\int_0^1 2x^2 - 3xdx = 2x^3/3 - 3x^2/2|_0^1 = 2/3 - 3/2 = -5/6.$

(g) $\int_0^2 xe^{x^2}dx$

Solution: $\int_0^2 xe^{x^2}dx = e^{x^2}/2|_0^2 = e^4/2 - e^0/2 = (e^4 - 1)/2.$

(h) Find the derivative of $g(x) = x \ln x$. Evaluate $\int_1^e \ln xdx$.

Solution: Note that $g'(x) = \ln x + 1$, which is close to the function we want to antidifferentiate. Thus $\int_1^e \ln xdx = \int_1^e \ln x + 1 - 1dx = \int_1^e \ln x + 1 - \int_1^e 1dx = x \ln x - x|_1^e = (e - e) - (0 - 1) = 0 + 1 = 1.$