December 12, 2006 **Name** ________________________________

The total number of points available is 192. Throughout this test, **show your work.**

1. (15 points) Consider the function $f(x) = x^2 e^x$.

   (a) Find the values of x at which the line tangent to the graph of f is horizontal.

   Solution: $f'(x) = 2xe^x + x^2e^x = (x^2 + 2x)e^x = x(x+2)e^x$ so f has horizontal tangent lines (precisely where $f'(x) = 0$) at $x = 0$ and $x = -2$.

   (b) Find the intervals over which the function f is increasing.

   Solution: Constructing the sign chart for f', we see that f is increasing over $(-\infty, -2)$ and over $(0, \infty)$.

2. (30 points) Suppose $u(x)$ is a function whose derivative is

$$u'(x) = (x^2 - 1)(x - 3)^2(x - 5)(3x + 13).$$

   What this says is that u has already been differentiated and the function given is $u'(x)$. Recall that an important theorem tells you the intervals over which $u(x)$ is increasing based on $u'(x)$.

   (a) Find the critical points of $u(x)$.

   Solution: $x = -1, x = 1, x = 3, x = 5$ and $x = -13/3$.

   (b) Use the Test Interval Technique to find the intervals over which $u'(x)$ is positive.

   Solution: The sign chart for u' shows that u' is positive over each of the intervals $(-\infty, -13/3), (-1, 1)$, and $(5, \infty)$.

   (c) Use the information in part (b) to find the intervals over which $u(x)$ is increasing.

   Solution: u is increasing over $(-\infty, -13/3), (-1, 1)$ and over $(5, \infty)$.

3. (25 points) Consider the function $f(x) = 36x^3 - 21x^2 + 12x - 7$.

   (a) Find an antiderivative of $f(x)$.

   Solution: One antiderivative is $y = 9x^4 - 7x^3 + 6x^2 - 7x$.

   (b) Find an antiderivative of $f(x)$ whose value at $x = 3$ is 2.

   Solution: Let $F(x) = 9x^4 - 7x^3 + 6x^2 - 7x + C$. Then $F(3) = 2 = 9 \cdot 3^4 - 7 \cdot 3^3 + 6 \cdot 3^2 - 7 \cdot 3 + C$, and it follows that $C = -571$, so $F(x) = 9x^4 - 7x^3 + 6x^2 - 7x - 571$.

(c) Compute $\int_0^1 f(x)$.

Solution: $\int_0^1 f(x) = F(1) - F(0) = 9 - 7 + 6 - 7 = 1$.

4. (20 points) Consider the function $f(x) = (x^2 - 4)^{2/3}$.

(a) Find $f'(x)$ and $f''(x)$. Discuss their domains.

Solution: $f'(x) = 4x(x^2 - 4)^{-1/3}/3$ and $f''(x) = 4(x^2 - 4)^{-1/3}/3 - 8x^2(x^2 - 4)^{-4/3}/9$.

(b) Find the critical points of f and identify them as stationary or singular.

Solution: $x = 0$ is a stationary point and $x = \pm 2$ are singular points.

(c) Use the sign chart for f' to decide, for each critical point, whether a local maximum, local minimum, or neither occurs at that critical point.

Solution: The sign chart for f' shows that f has a local maximum at $x = 0$ and local minimums at both 2 and -2.

5. (20 points) Let $f(x) = \dfrac{4}{x} - 2e^x$.

   (a) Find an antiderivative of $f(x)$.
   Solution: Note that $\int \frac{4}{x} - 2e^x.\ dx = 4\ln x - 2e^x + C$.

   (b) Find an antiderivative of $f(x)$ with a value of 4 at the point $x = 1$.
   Solution: Let $F(x) = \int \frac{4}{x} - 2e^x.\ dx = 4\ln x - 2e^x + C$. Then $F(1) = 4 = 4\ln 1 - 2e^1 + C$ and so $C = 4 + 2e$.

   (c) Compute $\displaystyle\int_1^4 f(x)$.
   Solution: We measure the growth of the antiderivative $4\ln x - 2e^x$ from 1 to 4: we get $4\ln x - 2e^x|_1^4 = 4\ln 4 - 2e^4 - (4\ln 1 - 2e) = 4\ln 4 - 2e^4 + 2e \approx -98.214$.

6. (10 points) What is the value of $\displaystyle\int_0^4 2x\sqrt{x^2+1}\ dx$?

   Solution: Use substitution to find an antiderivative. Let $u = x^2 + 1$. Then $du = 2x\ dx$. Now the integral becomes $\displaystyle\int 2x\sqrt{x^2+1}\ dx = \int u^{1/2}\ du = 2u^{3/2}/3$. We must replace the function with its x equivalent and measure its growth from 0 to 4. We get $2(x^2+1)^{3/2}/3|_0^4 = 2(17^{3/2} - 1^{3/2})/3 \approx 46.062$.

7. (15 points) Compound Interest.

   (a) Consider the equation $1000(1 + 0.01)^{12t} = 3000$. Find the value of t and interpret your answer in the language of compound interest. In other words, write a sentence about what the value of t is.

   Solution: t is the tripling time for a 12% investment compounded monthly. Use logs to solve for t to get $t \approx 9.201$ years.

   (b) Consider the equation $P(1 + 0.03)^{4\cdot 10} = 3000$. Solve for P and interpret your answer in the language of compound interest.

   Solution: What is the present value of 3000 in ten years at 12% compounded quarterly OR in 4 years at 30% compounded 10 times per year. $P \approx 919.67$ dollars.

   (c) Consider the equation $1000e^{8r} = 2000$. Solve for r and interpret your answer in the language of compound interest.

   Solution: In order to double your money in 8 years, you must earn $r = 8.66\%$ when interest is compounded continuously.

8. (10 points) Find the value of $\displaystyle\int_0^{\sqrt{\ln 6}} \frac{de^{x^2}}{dx}$. Hint: This problem is much easier than it looks.

   Solution: An antiderivative of $\frac{de^{x^2}}{dx}$ is e^{x^2}, so we need only measure the growth of e^{x^2} over the interval 0 to $\sqrt{\ln 6}$. This is just $e^{\ln 6} - e^0 = 6 - 1 = 5$.

9. (20 points) A manufacture has been selling 1300 television sets a week at \$450 each. A market survey indicates that for each \$20 rebate offered to a buyer, the number of sets sold will increase by 270 per week. In other words, if they drop the price by \$20, they sell 270 more sets, etc.

   (a) Find the demand function $p(x)$, where x is the number of the television sets sold per week, and $p(x)$ is measured in dollars.

   Solution: First find two values of the linear function $p(x)$. Note that $p(1300) = 450$ and $p(1570) = 430$ so we can find $p(x)$ using the two-point form of the line: $y - 450 = -\frac{2}{27}(x - 1300)$, which, after some approximating, yields $y \approx -\frac{2x}{27} + 546.29$.

   (b) How large rebate should the company offer to a buyer, in order to maximize its revenue?

   Solution: We want to maximize $R(x) = xp(x)$, so find $R'(x) = p(x) + xp'(x) \approx -\frac{4x}{27} + 546.3$. The critical point is $x = 3687.5$ which corresponds to a price of $p(3687.5) = 273.14$. So the rebate is for \$176.85

   (c) If the weekly cost function is $97500 + 150x$, how should it set the size of the rebate to maximize its profit?

   Solution: Since Profit is Revenue minus Cost, we have $P(x) = R(x) - C(x)$ and $P'(x) = xp'(x) + p(x) - C'(x) = x(-\frac{2}{27}) + 546.3 - \frac{2x}{27} - 150 = -\frac{4x}{27} + 396.3$ from which it follows that $x = 2675.03$ and $p(2675.03) = 348.15$, the optimal price for profit. So the rebate is for \$101.85

10. (12 points) The population of the world in 1987 was 5 billion and the relative growth rate was estimated at 1.4 percent per year. Assuming that the world population follows an exponential growth model, find the projected world population in 1995.

    Solution: First, $P(t) = P_0 e^{kt}$ is the growth model, with Q's changed to P's for clarity. Since $P(0) = 5$ billion, it follows that $P_0 = 5$ and $k = 0.014$. Since 1995 is 8 years after 1987, we need $P(8)$. $P(8) = 5e^{0.014 \cdot 8} = 5(1.1185128) = 5.592$ billion people.

11. (15 points) Certain radioactive material decays in such a way that the mass remaining after t years is given by the function

$$m(t) = 135e^{-0.01t}$$

where $m(t)$ is measured in grams.

(a) Find the mass at time $t = 0$.
Solution: $m(0) = 135e^0 = 135$ grams.

(b) How much of the mass remains after 10 years?
Solution: After 10 years we have $m(10) = 135e^{-.01 \cdot 10} \approx 122.153$ grams.

(c) What is the half-life of the material?
Solution: We need to solve $e^{-0.01t} = .5$ for the half-life t. Taking ln of both sides, we get $t = \ln 0.5/-0.01 \approx 69.3147$ years.