December 11, 2007 **Name** ______________________________

The total number of points available is 238. Throughout this test, **show your work.**

1. (15 points) Consider the function $f(x) = xe^{2x}$.

   (a) Find a value of x at which the line tangent to the graph of f is horizontal.

   Solution: Since $f'(x) = e^{2x} + 2xe^{2x}$, we can factor and solve $e^{2x}(1+2x) = 0$ to get $x = -1/2$.

   (b) Find a value of x at which the line tangent to the graph of f has slope $2e$.

   Solution: Setting $e^{2x}(1 + 2x) = 2e$, we see that $x = 1/2$ works.

   (c) Find an equation of the tangent line referred to in part (b).

   Solution: Note that $f(1/2) = e/2$, so the line must be $y - e/2 = 2e(x - 1/2)$. Thus $y \approx 5.437x - 1.359$

2. (30 points) Suppose $u(x)$ is a function whose derivative is

$$u'(x) = (x^2 - 9)(x - 1)^2(x - 7)(3x + 11).$$

   What this says is that u has already been differentiated and the function given is $u'(x)$ Recall that an important theorem tells you the intervals over which $u(x)$ is increasing based on $u'(x)$.

   (a) Find the critical points of $u(x)$.

   Solution: $x = -3, x = 3, x = 1, x = 7$ and $x = -11/3$.

   (b) Use the Test Interval Technique to find the intervals over which $u(x)$ is increasing.

   Solution: u is increasing over $(-\infty, -11/3), (-3, 3)$ and over $(7, \infty)$.

3. (12 points) Given $f''(x) = 2x - 6$ and $f'(-2) = 6$ and $f(-2) = 0$. Find $f'(x)$ and $f(x)$.

   Solution: First write $f'(x) = x^2 - 6x + C$ by the power rule. Solve $f'(-2) = 6$ for C to get $C = -10$. Then $f(x) = x^2 - 6x - 10$. Therefore $f(x) = \int x^2 - 6x - 10\, dx = x^3/3 - 3x^2 - 10x + C$. We can solve $f(-2) = 0$ to get $C - 16/3$, so the function is $f(x) = x^3/3 - 3x^2 - 10x - 16/3$.

4. (15 points) Compound Interest.

   (a) Consider the equation $1000(1+0.02)^{4t} = 5000$. Find the value of t and interpret your answer in the language of compound interest.

   Solution: t is the time required for an 8% investment compounded quarterly to quintuple. To find t, solve $4t \ln 1.02 = \ln 5$, getting $t \approx 20.318$ years.

   (b) Consider the equation $P(1+0.03)^{4\cdot 10} = 5000$. Solve for P and interpret your answer in the language of compound interest.

   Solution: P is the present value of \$5000 invested at 12% compounded quarterly for 10 years. Solve the equation to get $P = 1532.78$.

   (c) Consider the equation $1000e^{10r} = 5000$. Solve for r and interpret your answer in the language of compound interest.

   Solution: What rate of interest does it take to quintuple an investment compounded continuously for 10 years. The value is $r = 16.09\%$.

5. (12 points) Let $f(x) = \dfrac{6}{x} - 2e^x$.

   (a) Find an antiderivative of $f(x)$.

   Solution: Note that $\int \frac{6}{x} - 2e^x \, dx = 6\ln x - 2e^x$.

   (b) Compute $\displaystyle\int_1^e f(x)dx$.

   Solution: Note that $\int \frac{6}{x} - 2e^x \, dx = 6\ln x - 2e^x$. So $\displaystyle\int_1^e f(x)dx = 6\ln x - 2e^x\Big|_1^e = 6\ln e - 2e^e - (6\ln 1 - 2e) = 6 - 2e^e + 2e \approx -18.872$.

6. (42 points) Find the following antiderivatives.

   (a) $\displaystyle\int 4x - 5 \, dx$

   Solution: $2x^2 - 5x + C$.

   (b) $\displaystyle\int 9x^2 - 4x - 1/x \, dx$

   Solution: $3 \cdot x^3 - 2 \cdot x^2 - \ln x + C$.

   (c) $\displaystyle\int \frac{x^3 + 2x^2 - x}{x} \, dx$

   Solution: $\int (x^3 + 2x^2 - x)/x \, dx = \int x^2 + 2x - 1 dx = x^3/3 + x^2 - x + C$.

(d) $\displaystyle\int \frac{2x+3}{x^2+3x-3}\,dx$

Solution: By substitution, $(u = x^2+3x-3)$, $\int \frac{2x+3}{x^2+3x-3}\,dx = \ln|x^2+3x-3| + C$.

(e) $\displaystyle\int 6x^5(x^6+3)^7\,dx$

Solution: By substitution with $u = x^6+3$, $\int 6x^4(x^6+3)^7\,dx = \frac{(x^6+3)^8}{8} + C$.

(f) $\displaystyle\int x^2 e^{x^3}\,dx$

Solution: By substitution with $u = x^3, du = 3x^2$, $\int e^u\,du = e^u + C = (1/3)e^{x^3} + C$.

7. (40 points) This question is about building more complicated functions from simpler ones. Let $f(x) = x^2$, $g(x) = \sqrt{x}$, $h(x) = x+1$, $k(x) = 1/x$ and $l(x) = x-2$. For each function given below, show how it is possible to combine some of the simpler functions above to obtain the given one. For example, if $U(x) = \sqrt{x^2-2}$ was given, you could write $U(x) = g \circ l \circ f(x)$, and if $V(x) = ((x+1)/x)^2$, you could write $V(x) = f \circ (h \cdot k)(x)$.

(a) $H(x) = \left(\frac{1}{x-2}\right)^2 + 1$

Solution: $H(x) = h \circ f \circ k \circ l(x)$.

(b) $G(x) = \left(\frac{1}{x-2} + 1\right)^2$

Solution: $G(x) = f \circ h \circ k \circ l(x)$.

(c) $L(x) = \frac{x}{x-2} - 2$

Solution: $L(x) = l(f \cdot k(k \circ l))(x)$.

(d) $K(x) = \frac{1}{(x+1)^2-2}$

Solution: $K(x) = k \circ l \circ f \circ h(x)$.

(e) $N(x) = (\sqrt{x^2+x+1} - 2)^2$

Solution: $N(x) = f \circ l \circ g \circ (f + h(x))$. There are probably other solutions as well.

8. (15 points) Find the intervals over which $f(x) = x^2 e^{2x}$ is increasing.

Solution: First $f'(x) = 2xe^{2x} + x^2 \cdot 2e^{2x} = 2e^{2x}(x+x^2)$, so we need to solve $x + x^2 = 0$ and we get $x = 0$ and $x = -1$. Since f' is negative precisely on $(-1, 0)$, f is increasing on $(-\infty, -1)$ and on $(0, \infty)$.

9. (12 points) Is there a value of b for which $\int_{b}^{2b} x^4 + x^2\,dx = 128/15$? If so, find it.

   Solution: Use the power rule to get the equation $(2b)^5/5+(2b)^3-(b^5/5+b) = 128/15$. It follows that $b = 1$.

10. (25 points) According to Newton's Law of Cooling, the rate at which an object's temperature changes is proportional to the temperature of the medium into which it is emersed. If $F(t)$ denotes the temperature of a cup of instant coffee (initially $212°F$), then it can be proven that

$$F(t) = T + Ae^{-kt},$$

   where T is the air temperature, $70°F$, A and k are constants, and t is expressed in minutes.

   (a) What is the value of A?

   Solution: Note that $F(0) = 70 + A \cdot 1 = 212$ so $A = 142$.

   (b) Suppose that after exactly 20 minutes, the temperature of the coffee is $186.6°F$. What is the value of k?

   Solution: Solve $F(t) = 186.6 = 70+142e^{-k(20)}$ for k to get $k \approx 0.009853$.

   (c) Use the information in (a) and (b) to find the number of minutes before the coffee reaches the temperature of $80°F$.

   Solution: Solve the equation $80 = 70 + 142e^{-0.009853t}$ for t to get first $e^{-0.009853t} = 10/142 \approx 0.0704$, and taking logs of both sides yields $t = 269.28$ minutes.

   (d) Find the rate at which the object is cooling after $t = 20$ minutes.

   Solution: To find $F'(t)$ recall the way we differentiate exponential functions. $F'(t) = 142(-k)e^{-kt}$, so $F'(20) = 140(-k)e^{-20k} \approx -1.1488$ degrees per minute.

11. (20 points) Amber Airlines runs chartered flights to Costa Rica. They expect 300 passengers and they charge each passenger \$200. However if more than 300 persons sign up for the flight, they agree to charge \$0.25 less per ticket for each extra person. For example, if 302 passengers sign up, the airline charges each of the 302 passengers \$199.50.

    (a) Find the revenue function $R(x)$ in terms of the number of new passengers x. In other words, let $x + 300$ represent the number of passengers, where $x > 0$.

    Solution: Let x represent the number of passengers beyond 300 that Amber Airlines enlists. Then $R(x) = (300 + x)(200 - 0.25x)$.

    (b) How many passengers result in the maximum revenue?

    Solution: To maximize $R(x)$, find the critical points and the endpoints of the domain. The domain is $[0, \infty)$, and, by the product rule, the derivative is $R'(x) = 1(200-.25)-0.25(300+x) = 200-.25x-75-.25x = 125-0.5x$. So $x = 250$ is the only critical point. Note that $R''(x) = -0.5$, so $R''(250) = -1.5 < 0$, and this means that $x = 250$ is the location of a relative maximum. Since $R(0) = 200 \cdot 300 = 60000$ is the only endpoint, and since R is decreasing to the right of $x = 250$ (why?, $R'(x)$ is negative for $x > 250$), it follows that R has an absolute maximum at $x = 250$.

    (c) What is that maximum revenue?

    Solution: The maximum revenue is $R(250) = 550 \cdot 137.50 = 75625$.