December 14, 2010 **Name** ______________________________

The total number of points available is 232. Throughout this test, **show your work. Throughout this test, you are expected to use calculus to solve problems. Graphing calculator solutions will generally be worth substantially less credit. Circle your answer in each of the three multiple choice problems.** This exam is 7 pages long.

1. The function f is defined throughout $[-1, 1]$. Suppose $lim_{x\to 0} f(x) = 3$, what is $f(0)$?

   (A) 0 **(B)** 3 **(C)** It must be close to 3

   (D) $f(0)$ is not defined **(E)** Not enough information is given

   Solution: E. Using the blotter test, recall that the limit does not depend on the value of the function at the point, so $f(0)$ can be anything. In other words, not enough information is given.

2. If f is a differentiable increasing function and f is (choose one), then the relationship $f'(5) \leq \frac{f(5)-f(2)}{5-2} \leq f'(2)$ is certain to hold.

   (A) Positive **(B)** Negative **(C)** Concave up

   (D) Concave down **(E)** There is no way to guarantee the relationship.

   Solution: D. If f is concave downward, over the interval $[2, 5]$, it must satisfy the inequality. None of the other conditions guarantee the inequality.

3. If a function is always positive, then what must be true about its derivative?

   (A) The derivative is always positive.

   (B) The derivative is never negative.

   (C) The derivative is increasing.

   (D) The derivative is decreasing.

   (E) You can't conclude anything about the derivative.

   Solution: E. You cannot draw any conclusions about the derivative.

4. (12 points) Find an equation for the line tangent to the graph of $f(x) = (1 + e^{-2x+4})^2$ at the point $(2, f(2))$.

   Solution: Find f' first: $f'(x) = 2(1 + e^{-2x+4})(e^{-2x+4})(-2)$. Then note that $f'(2) = 2(1 + e^0)(e^0)(-2) = -8$ and $f(2) = 4$, so the line is $y = -8x + 20$.

5. (12 points) Find an equation for the line tangent to the graph of $f(x) = (x^2 + \ln(x))^2$ at the point $(1, f(1))$.

   Solution: First $f'(x) = 2(x^2 + \ln(x))(2x + 1/x)$. Then note that $f'(1) = 2(1+0)(3) = 6$, so the line is $y = 6x - 5$.

6. (12 points) A radioactive substance has a half-life of 29 years. Find an expression for the amount of the substance at time t if 30 grams were present initially.

   Solution: $Q(t) = Q_0 e^{-kt}$. Since $Q_0 = 30$ and the half-life is 29 years, it follows that $15 = 30e^{-29k}$, which can be solved to give $k \approx 0.0239$. Thus $Q(t) = 30e^{-0.0239t}$.

7. (12 points) Build the sign chart for the rational function

$$r(x) = \frac{(x-2)^2 \cdot x(x+1)}{(x^2-4)(2x-7)^2}.$$

   Clearly label the branch points.

   Solution: After cancelling common terms, we have $r(x) = \frac{(x-2)x(x+1)}{(x+2)(2x-7)^2}$. This function can change signs only at $x = 2, x = -2, x = 7/2, x = 0$, and $x = -1$. Note however the squared factor $(2x-7)^2$, which means there is no change of sign at $x = 7/2$. Hence, the function is positive on the intervals $(-\infty, -2), (-1, 0), (2, 7/2)$ and $(7/2, \infty)$.

8. (12 points) For what value(s) of x is the line tangent to $y = 4 - x^2$ parallel to the line $y = x$.

   Solution: We want to know where $y' = -2x$ is equal to the slope of the line $y = x$, which is 1. Solve $-2x = 1$ to get $x = -1/2$.

9. (12 points) Let $H(x) = \frac{1}{x-3} + \frac{1}{x^2-4}$. Is this a rational function? Find all the asymptotes.

   Solution: Yes, its rational. Find a common denominator and add to get $H(x) = \frac{x^2+x-7}{(x-3)(x-2)(x+2)}$, so it has horizontal asymptote $y = 0$ and three vertical asymptotes, $x = 3, x = 2, x = -2$.

10. (16 points) Let $A = (2, 0)$, $B = (10, 0)$, $C = (10, 12)$ and $D = (2, 6)$. The area of the quadrilateral $ABCD$ is 72.

    (a) Find an equation for the linear function (the line) that goes through the points C and D. Give this function the name f.

    Solution: Since we know two points on the line, it follows that $f(x)-6 = \frac{12-6}{10-2}(x-2)$, which we can write as $f(x) = 3x/4 + 9/2$.

    (b) Use calculus to find the area of the region R defined as follows:

    $$R = \{(x, y) : 2 \leq x \leq 10,\ 0 \leq y \leq f(x)\}$$

    Solution: We need to find $\int_2^{10} 3x/4 + 9/2\, dx$. This is just $\frac{3x^2}{8} + \frac{9x}{2}\Big|_2^{10} =$ 72.

11. (12 points) Find the derivatives of each function. Then find the slope of the line tangent to the graph of f at the point $(1, f(1))$.

    (a) $f(x) = (x-1)^2 \cdot \ln(2x+1)$.
    Solution: Using the product rule, $f'(x) = 2(x-1)\cdot\ln(2x+1) + \frac{2}{2x+1}(x-1)^2$, so $f'(1) = 0$.

    (b) $f(x) = \frac{\ln(2x+1))}{x}$.
    Solution: Using the quotient rule, $f'(x) = \frac{\frac{2}{2x+1}x - 1\cdot\ln(2x+1)}{x^2}$. Finally, $f'(1) = 2/3 - \ln(3)$.

12. (16 points) Suppose $f(x)$ is a function whose second derivative is constant. In fact, $f''(x) = 2$. Also suppose $f'(1) = 5$ and $f(1) = 4$.

    (a) Find $f'(0)$.
    Solution: $f'(x) = 2x + C$ for some constant C. Since $f'(1) = 2\cdot 1 + C = 5$, it follows that $C = 3$ and $f'(0) = 3$.

    (b) Find $f(0)$.
    Solution: Since $f(x) = x^2 + 3x + D$ for some constant D, and $f(1) = 1 + 3 + D = 4$, it follows that $D = 0$, $f(0) = 0$ and $f(x) = x^2 + 3x$.

    (c) Find the critical point(s) of f.
    Solution: Solve $f'(x) = 2x + 3 = 0$ to get the one critical point $x = -3/2$.

13. (36 points) Evaluate each of the following integrals using the Fundamental Theorem of Calculus (ie, antidifferentiate, then measure the growth of an antiderivative over the interval).

    (a) Evaluate $\displaystyle\int_1^3 \frac{d}{dx}(x^2 + 8x - 2)\ dx$

    Solution: We simply measure the growth of the antiderivative x^2+8x-2 over the interval $[1, 3]$: $\int_0^3 \frac{d}{dx}x^2 + 8x - 2\ dx = x^2 + 8x - 2|_1^3 = 9 + 24 - 2 - (1 + 8 - 2) = 31 - 7 = 24.$

    (b) Evaluate $\displaystyle\int_0^{\sqrt{\ln(2)}} \frac{d}{dx}e^{x^2}\ dx$

    Solution: We simply measure the growth of the antiderivative e^{x^2} over the interval $[0, \sqrt{\ln(2)}]$: $e^{(\sqrt{\ln(2)})^2} - e^0 = 2 - 1 = 1.$

    (c) Evaluate $\displaystyle\int_1^3 x^3 \cdot (x^4 - 2)^2\ dx$

    Solution: Let $u = x^4 - 2$. Then $du = 4x^3dx$ and $\int_1^3 x^3 \cdot (x^4 - 2)^2\ dx = \frac{1}{4}(x^4 - 2)^3\big|_1^3 = \frac{79^3}{12} - \frac{-1}{12} = 41086.5.$

    (d) Evaluate $\displaystyle\int_0^4 3x^2e^{x^3}\ dx$

    Solution: $\int_0^4 3x^2e^{x^3}\ dx = e^{x^3}|_0^4 = e^{64} - e^0 \approx 6.23 \cdot 10^{27}.$

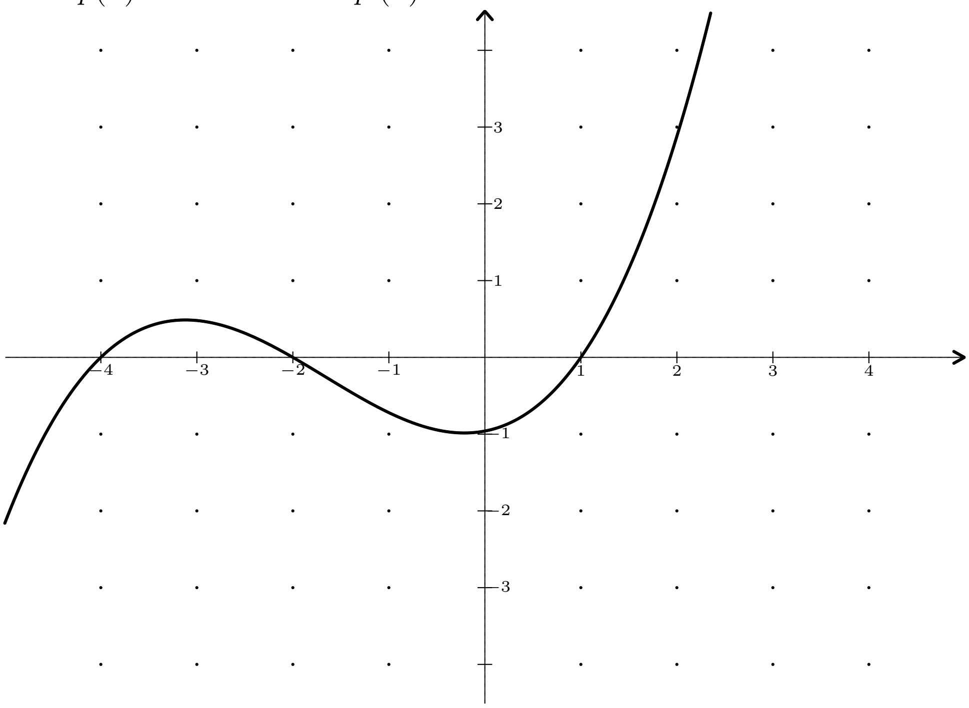
14. (10 points) Evaluate $\int x^2 - \sqrt{x} - \frac{2}{x}\,dx$

    Solution: $\int x^2 - \sqrt{x} - \frac{1}{x}\,dx = \int x^2\,dx - \int \sqrt{x}\,dx - \int -\frac{1}{x}\,dx = x^3/3 - 2x^{3/2}/3 + \ln|x| + C$.

15. (10 points) Evaluate $\int 3x^2\sqrt{x^3+4}\,dx$

    Solution: Use substitution with $u = x^3 + 4$ to get $1/3\int u^{1/2}\,du = 2u^{3/2}/9 = 2(x^3+4)^{3/2}/9 + C$.

16. (20 points) The graph of a cubic polynomial $p(x)$ is given below. On the same set of axes, sketch the graph of $p'(x)$, being especially careful about where $p(x) = 0$ and where $p'(x) = 0$.



Solution: The derivative is a quadratic function opening upwards with zeros at $x \approx -3$ and $x \approx 0$.

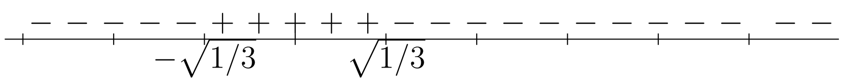
17. (25 points) Consider the function $f(x) = \ln(3x^2 + 1)$.

    (a) Find $f'(x)$.
    Solution: $f'(x) = \frac{6x}{3x^2+1}$.

    (b) Find an equation for the line tangent to the graph of f at the point $(3, f(3))$.
    Solution: Since $f'(3) = 18/28 = 9/14$ and $f(3) = \ln 28$, we have $y - \ln 28 = 9(x - 3)/14$.

    (c) Find $f''(x)$.
    Solution: $f''(x) = \frac{6(3x^2+1)-6x(6x)}{(3x^2+1)^2}$.

    (d) Find the sign chart for $f''(x)$.
    Solution: $f''(x) < 0$ on $(-\infty, -\sqrt{1/3})$ and on $(\sqrt{1/3}, \infty)$ and positive on $(-\sqrt{1/3}, \sqrt{1/3})$, as shown on the sign chart for f'':



    (e) Find the intervals over which f is concave upwards.
    Solution: From (c) it follows that f is concave upwards on $(-\sqrt{1/3}, \sqrt{1/3})$.