May 11, 2011 **Name** ________________________________

The total number of points available is 305. Throughout the free response part of this test, **show your work.** Each of the first 20 problems is worth 10 points.

1. Let $f(x) = x^4 - 2x + 4$. What is $f'(1)$?

   Solution: Since $f'(x) = 4x^3 - 2$, so $f'(1) = 4 - 2 = 2$.

2. Find an equation for the line tangent to the graph of $f(x) = 3x^3 - 2x + 4$ at the point $(2, f(2))$?

   Solution: Since $f'(x) = 9x^2 - 2$, $f'(2) = 34$, so the tangent line is $y - 24 = 36(x - 2)$.

3. Consider the function $f(x) = (e^{2x} + 1)^3$. What is the slope of line tangent to the graph of f at the point $(1, f(1))$?

   Solution: Since $f'(x) = 3(e^{2x} + 1)^2 \cdot (2e^{2x})$ by the chain rule, $f'(1) = 3(e^2 + 1)^2 \cdot 2e^2 \approx 3120.1$.

4. Suppose the line $3x + 4y = 11$ is tangent to the graph of $h(x)$ at the point $(1, 2)$. What is $h'(1)$?

   Solution: The slope of the line is $m = -3/4$.

5. What is $\lim_{x\to\infty} \dfrac{(3x+2)(4x-1)}{(x-2)(2x-3)}$?

   Solution: Using the asymptote theorem, $\lim_{x\to\infty} \dfrac{(3x+2)(4x-1)}{(x-2)(2x-3)} = 12/2 = 6$.

6. What is the exact value of $|2\pi - 7| + |8 - 2\pi| + \pi$? Leave your answer in terms of π. No credit for a decimal approximation.

   Solution: The value is $7 - 2\pi + 8 - 2\pi + \pi = 15 - 3\pi$.

7. What is $\lim_{x\to 2} \dfrac{x^2-4}{x^3-8}$?

   Solution: Factor both numerator and denominator to get $\lim_{x\to 2} \frac{x^2-4}{x^3-8} = \lim_{x\to 2} \frac{(x-2)(x+2)}{(x-2)(x^2+2x+4)} = 4/12 = 1/3$

8. Find a function f that satisfies (a) $f'(x) = 3x^2 - 2x$ and (b) $f(2) = 3$.

   Solution: Antidifferentiate to get $f(x) = x^3 - x^2 + C$. Thus $f(2) = 8 - 4 + C = 3$, so $C = -1$ and $f(x) = x^3 - x^2 - 1$.

9. Let $H(x) = \ln(4x^2 + 12x + 10) - 2x$. Find all critical points of H.

   Solution: We need to solve the equation $\frac{8x+12}{4x^2+12x+10} = 2$. This is equivalent to $8x^2 + 16x + 8 = 0$ which has repeated roots, $x = -1$

10. Let $g(x) = 2x^3 - 7x^2 + 4x - 10$. Find the intervals over which g is decreasing?

    Solution: Note that $g'(x) = 6x^2 - 14x + 4$. Build the sign chart for $g'(x)$ to find that $g'(x) \leq 0$ on $[1/3, 2]$.

11. Let $k(x) = 2x^4 - 14x^3 + 30x^2 + 10x$. Over which intervals is k is concave upwards?

    Solution: $k''(x) = 12(2x - 5)(x - 1) > 0$ on $(-\infty, 1)$ and $(5/2, \infty)$.

12. What is the value of $\int_2^4 \frac{d}{dx}(3x - 5)^4 \, dx$

    Solution: Since differentiation and antidifferentiation just undo each other, its just $(3x - 5)^4|_2^4 = 7^4 - (1)^4 = 2401 - 1 = 2400$.

13. What is the area of the region R bounded above by $y = 2x + 1$, below by $y = x - 7$, on the left by $x = 2$ and on the right by $x = 4$?

    Solution: Let $f(x) = 2x + 1 - (x - 7) = x + 4$. Now $\int_2^4 x + 8 = x^2/2 + 8x|_2^4 = 8 + 32 - (2 + 16) = 22$.

14. Find a value of b for which $\displaystyle\int_b^{2b} \frac{1}{x} + 1\, dx = \ln(2) + 6$.

    Solution: Solve $\ln(2b) + 2b - \ln(b) - b = \ln(2) + b = \ln(2) + 6$ for b to get $b = 6$.

15. What is the absolute maximum value of the function $f(x) = 2x^3 - 9x^2 + 12x + 5$ on the interval $-2 \le x \le 3$?

    Solution: Find $f'(x)$ first and then the critical points that are between -2 and 3. $f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$, so there are two critical points and two endpoints to check. $f(-2) = -71$; $f(1) = 10$; $f(2) = 9$; and $f(3) = 14$, so the absolute maximum is $f(3) = 14$, which of course occurs at $x = 3$.

16. Find all the zeros of the polynomial $p(x) = (x - 1)^3(x + 2)^2 - 4(x - 1)^2(x + 2)$.

    Solution: Factor p to get $p(x) = (x-1)^2(x+2)[(x-1)(x+2)-4]$. Two zeros are $x = 1$ and $x = -2$ and the other two are the solutions to $(x-1)(x+2)-4 = 0$, which yields $x^2 + x - 6 = (x + 3)(x - 2) = 0$ and $x = -3$ and $x = 2$.

17. Use calculus to find $\int e^{2x}(e^{2x} + 1)^4\ dx$.

    Solution: By substitution with $u = e^{2x} + 1$, we have $du = 2x^{2x}$ and then get the antiderivative $\frac{1}{2}\frac{u^5}{5} = \frac{1}{10}(e^{2x} + 1)^5 + C$.

18. Use calculus to find $\int \frac{2x}{x^2+1}\ dx$.

    Solution: Careful examination of the integrand reveals that it has the form $\frac{f'(x)}{f(x)}$, so $\int \frac{2x}{x^2+1}\ dx = \ln(x^2 + 1) + C$.

19. Use calculus to compute $\int_1^3 x^2 - x - \frac{1}{x} + 1\ dx$.

    Solution: Doing one piece at a time yields $\int_1^3 x^2 - x - \frac{1}{x} + 1\ dx = x^3/3 - x^2/2 - \ln(x) + x|_1^3 = 2/3 - \ln(3)$.

20. Given that the graph of f passes through the point $(1, 5)$ and that the slope of its tangent line at $(x, f(x))$ is $2x + 1$, what is $f(4)$?

    Solution: $f(x) = x^2 + x + c$ so $f(1) = 1 + 1 + c = 5$. It follows that $c = 3$. Thus, $f(4) = 4^2 + 4 + 3 = 23$.

21. (15 points) Rachel learns typing in a 14 week class. The number of words per minute Rachel can type after t weeks is given by

$$F(t) = 120 - 40e^{-0.4t}.$$

(a) How many more words per minute can Rachel type after the third week than she can type after the second week? (b) What is $F'(2.5)$? (c) How are these numbers related?

Solution: $F(3) - F(2) = 120 - 40e^{-1.2} - 120 - 40e^{-0.8} = 40(e^{-0.8} - e^{-1.2} \approx$ 5.9 words per minute. On the other hand, $F'(t) = 16e^{-0.4t}$, so $F'(2.5) = 16e^{-1} \approx 5.88$ words per minute. This is not surprising because to us because F' measures the rate of learning.

22. (20 points) Find the area of the region caught between the functions $f(x) = 5 - x^2$ and $g(x) = 2x - 3$. Show how you used the Fundamental Theorem by measuring the growth of an antiderivative over an interval. Your work must make clear what interval you used.

Solution: The area is $\int_{-4}^{2} 5 - x^2 - (2x + 3)\ dx = \int_{-4}^{2} -x^2 - 2x + 8)\ dx = -x^3/3 - x^2 + 8x|_{-4}^{2} = 60 - 72/3 = 36$.

23. (30 points) Let $h(x) = \frac{x(2x+11)(2x+7)}{(x-1)^2(3x-12)}$.

   (a) Find the asymptotes of h.

   Solution: Solve $x - 1 = 0$ to get $x = 1$ and solve $3x - 12 = 0$ to get $x = 4$ for asymptotes.

   (b) Find the zeros of h.

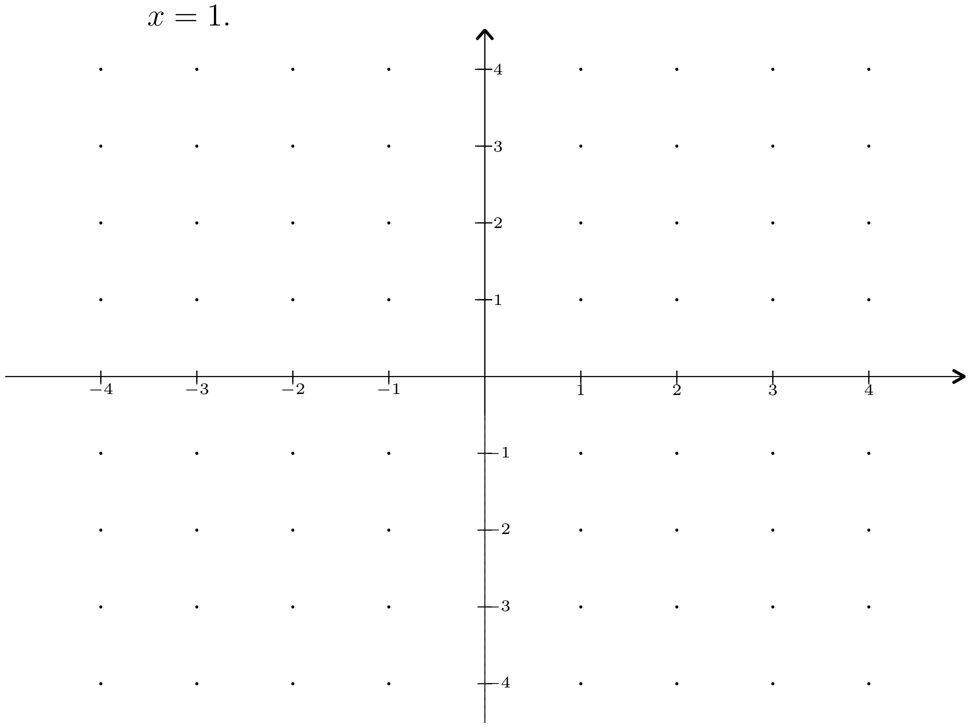
   Solution: Solve $2x+7 = 0$ to get the zero $x = -7/2$ and solve $2x+11 = 0$ to get the zero $x = -11/2$. Of course $x = 0$ is also a zero of h.

   (c) Build the sign chart for $h(x)$.

   Solution: The sign chart shows that h is positive over $(-\infty, -11/2)$, $(-7/2, 0)$ and $(4, \infty)$, and negative on the open intervals $(-11/2, -7/2)$, $(0, 1)$ and $(1, 4)$.

   (d) Sketch the graph of $h(x)$ USING the information in (a) and (b).

   Solution: The graph must show that there are relative extrema at two values, a minimum between -2 and -3 and a maximum between 1 and 2.5. Also, your graph must make clear that there is not sign change at $x = 1$.



24. (20 points) Let $H(x) = \sqrt{(3x+1)^{12}+3}$.

  (a) Find three functions f, g and h satisfying $f(g(h(x))) = f \circ g \circ h(x) = H(x)$.
  Solution: One way to do this is to let $f(x) = \sqrt{x}$, $g(x) = x^{12}+3$, and $h(x) = 3x+1$.

  (b) Compute the derivative of each of the three component functions f, g, h.
  Solution: In case we choose the functions above, we get $f'(x) = x^{-1/2}/2$, $g'(x) = 12x^{11}$, and $h'(x) = 3$.

  (c) Apply the chain rule twice to find $H'(x)$.
  Solution: $H'(x) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x) = \frac{1}{2}((3x+1)^{12}+3)^{-1/2} \cdot 12(3x+1)^{11} \cdot 3 = 18((3x+1)^{12}+3)^{-1/2} \cdot (3x+1)^{11}$.

25. (20 points) The quadrilateral T with vertices $A = (0,0)$, $B = (0,6)$, $C = (8,10)$ and $D = (8,0)$ is a trapezoid since the two sides AB and CD are both vertical. It is not hard to see that the area of T is 64 square units.

  (a) Find an equation for the line passing through the points B and C. Let $f(x)$ be the function whose graph is this line.
  Solution: The slope of the line is $m = \frac{10-6}{8-0} = \frac{1}{2}$, so the line is $y - 6 = \frac{1}{2}(x-0)$, which is $y = x/2+6$

  (b) Use calculus, showing all your work, to verify that the area of the region T bounded above by the graph of f, below by the x-axis, and on the sides by $x = 0$ and $x = 8$ is 64.
  Solution: Since the function is positive, the area is the same as the integral, $\int_0^8 x/2+6\ dx = x^2/4+6x|_0^8 = 64/4+6\cdot 8-0 = 16+48 = 64$.