Math 1120 **Calculus** **Final Exam, section 6**

December 13, 2011 Name ____________________

The total number of points available on this test is 273.

1. (10 points) Let $f(x) = x^2 - 3x + 5$. Find an equation for the line tangent (in slope-intercept form) to the graph of f at the point $(2, f(2))$.

   Solution: Note that $f'(x) = 2x - 3$, so $f'(2) = 1$. Also, $f(2) = 3$.Therefore the line in point-slope form is $y - 3 = 1(x - 2)$ which translates into $y = x + 1$.

2. (10 points) The line tangent to the graph of a function f at the point $(3, 5)$ on the graph also goes through the point $(0, 8)$. What is $f'(3)$?

   Solution: The slope of the line through $(3, 5)$ and $(0, 8)$ is $f'(3) = \frac{8-5}{0-3} = -1$.

3. (10 points) Suppose $f'(x) = 2x - \ln(x)$ and $f(e) = 3$. Find an equation for the line tangent to the graph of f at the point $(e, 3)$.

   Solution: $y - 3 = (2e - 1)(x - e)$.

4. (10 points) Build the sigh chart for the derivative of $g(x) = e^{x^2-4x}$ and use it to find the intervals over which g is increasing.

   Solution: Note that $g'(x) = (2x - 4)e^{x^2-4x}$ has only one zero, $x = 2$. The sign chart shows that g' is positive to the right of 2 and negative to the left of 2. So g is increasing on $(2, \infty)$.

5. (15 points) Let $f(x) = 2x^2 - 3x$.

   (a) Find a point $(a, f(a))$ on the graph of f where the tangent line has slope 0.

   Solution: The derivative function $f'(x) = 4x - 3$, which has value 0 when $4x - 3 = 0$, so $x = 3/4$ and $f(3/4) = 2 \cdot (3/4)^2 - 3 \cdot (3/4) = -9/8$.

   (b) Find a point $(a, f(a))$ on the graph of f where the tangent line has slope 1.

   Solution: $4x - 3 = 1$ precisely when $x = 1$. So the point on the graph is $(1, -1)$.

   (c) Find a point $(a, f(a))$ on the graph of f where the tangent line has slope 2.

   Solution: $4x - 3 = 1$ precisely when $x = 5/4$. So the point on the graph is $(5/4, -5/8)$.

6. (30 points) A manufacture has been selling 1850 television sets a week at \$510 each. A market survey indicates that for each \$10 rebate offered to a buyer, the number of sets sold will increase by 100 per week.

   (a) Find the linear demand function $p(x)$, where x is the number of the television sets sold per week.

   Solution: When the price goes down, the number of sets sold goes up, so the slope m is the linear demand curve is negative. The data shows that $m = -1/10$, so the function is $p(x) - 510 = -1(x - 1850)/10$ which is the same as $p(x) = -x/10 + 695$.

   (b) How large rebate should the company offer to a buyer, in order to maximize its **revenue**?

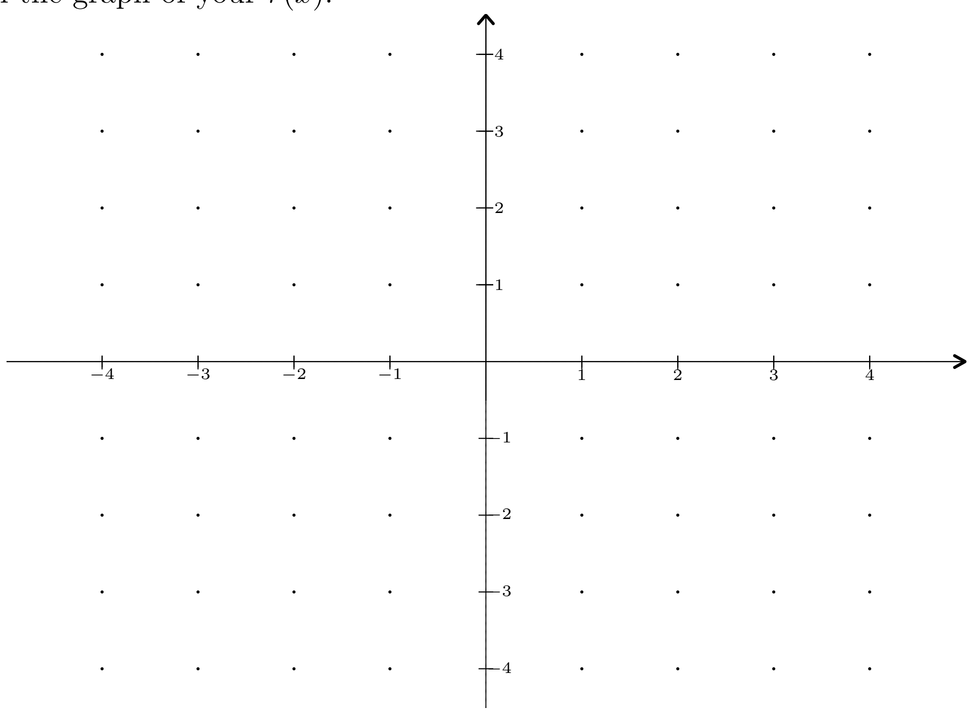
   Solution: $R(x)$ is the product of the price p and the number of sets sold x. $R(x) = xp(x) = -x^2/10 + 695x$. So $R'(x) = -x/5 + 695$, which has a zero at $x = 3475$. This means the rebate must be $(3475 - 1850)/10 =$ \$162.50. $R(3475) = 3475 \cdot (510 - 162.5) \approx 1.2 \cdot 10^6$.

   (c) If the weekly cost function is $157250 + 170x$, how should it set the size of the rebate to maximize its profit?

   Solution: The profit function $P(x) = R(x) - C(x)$ is given by $P(x) = -x^2/10 + 695x - (157250 + 170x) = -x^2/10 + 525x - 157250$. Thus, $P'(x) = -x/5 + 525$, which has 2625 as a zero. The rebate should be set so that the number of sets sold is 2625. The rebate is $(2625 - 1850)/10 =$ \$77.50.

7. (20 points) Find a rational function $r(x)$ that has exactly two zeros, $x = -2$ and $x = 1$, exactly two vertical asymptotes at $x = -1$ and $x = 3$, and has a horizontal asymptote $y = 2$.

   (a) Sketch the graph of your $r(x)$.



   Solution:

   (b) Find a symbolic representation of r.

   Solution: There are a few ways to do this. The easiest is to make the numerator $2(x+2)(x-1)$ and the denominator $(x-3)(x+1)$. To make the function have $y = 2$ as an asymptote, we can simply multiply by 2. Thus $r(x) = \frac{2(x+2)(x-1)}{(x-3)(x+1)}$.

   (c) Find the derivative of your function r and build the sign chart for $r'(x)$. Is there an interval over which r is increasing? If so, find it.

   Solution: By the quotient rule, $r'(x) = \frac{2(2x+1)(x^2-2x-3)-(2x-2)(2)(x^2+x-2)}{(x^2-2x-3)^2} = \frac{2(-3x^2-2x-7)}{(x^2-2x-3)^2}$, which is negative wherever it is defined. Therefore the function does not have any intervals over which it is increasing.

8. (12 points)

   (a) Find the rate of change of $f(x) = x^2 \ln(2x + 1)$ when $x = 1$.
   Solution: Use the product rule to get $f'(x) = 2x\ln(2x+1) + \frac{2x^2}{2x+1}$ whose value at $x = 1$ is $f'(1) = 2\ln(3) + (2/3)$.

   (b) Find the slope of the line tangent to f is the point $(2, 4\ln 5)$.
   Solution: Evaluate $f'(x)$ at $x = 2$ to get $f'(2) = 4\ln(5) + 8/5$.

9. (12 points) If we add a number to three times a second number, we get the answer 48. What is the largest possible product the two numbers can have?

   Solution: Let x denote the first number and y the second. Then $3x + y = 48$. Therefore $y = 48 - 3x$. The product of the two numbers is $P = xy = x(48-3x)$, so we can write $P(x) = 48x - 3x^2$ and $P'(x) = 48 - 6x$, so there is just one critical point, $x = 8$. This means $y = 24$ and the product is $P(8) = 8 \cdot 24 = 192$.

10. (12 points) An investment of \$1000 that is compound continuously takes exactly 11 years to double in value.

    (a) How long does it take to triple?

    Solution: The doubling requirement means that $2000 = 1000e^{11r}$, from which it follows that $r = \ln(2)/11 \approx 0.063$. Now solve $3 = e^{rt}$ for t to get $t = \ln(3)/r \approx 17.43$ years.

    (b) How long does it take to grow \$1000 into \$8000?

    Solution: The account must double three times, which takes 33 years.

    (c) Find a function that describes the amount in the account after t years?

    Solution: $A(t) = 1000e^{\ln(2)t/11}$.

    (d) Find the value of your function at $t = 33$ years.

    Solution: $A(33) = 1000e^{\ln(2)33/11} = 1000(e^{ln(2)})^3 = 1000 \cdot 2^3 = 8000$.

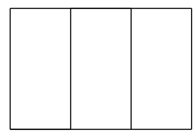
11. (12 points) If $h = g \circ f$ where $f(x) = x - 1/x,\ g(x) = x^2 + x + 2$. Find $h'(x)$. Then find $h'(1)$ and use that number to find an equation for the line tangent to the graph of $h(x)$ at the point $(1, 2)$.

    Solution: $h'(x) = g'(f(x)) \cdot f'(x) = (2(f(x)+1)(1+x^{-2}) = 2(x-1/x+1)(1+x^{-2})$, so $h'(1) = 2$. Therefore the line we seek is $y - 2 = 2(x - 1)$.

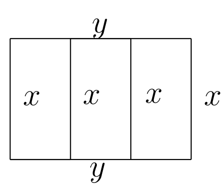
12. (15 points) Let $f(x) = x^2e^{3x}$. Find the interval(s) where f is concave upward.

    Solution: Use the product rule to get $f'(x) = 3e^{3x} \cdot x^2 + 2xe^{3x} = e^{3x}(3x^2 + 2x) = e^{3x}x(3x + 2)$ and $f''(x) = 3x^{3x}(3x^2 + 2x) + e^{3x}(6x + 2)$, which after massaging gets to be $e^{3x}(9x^2 + 12x + 2)$, which has two zeros, $\alpha = \frac{-2-\sqrt{2}}{3}$ and $\beta = \frac{-2+\sqrt{2}}{3}$. Therefore the function is convex upwards except between those two points, $(-\infty, \alpha)$ and (β, ∞).

13. (15 points) A rancher wants to fence in an area of 12 square miles in a rectangular field and then divide it into three pastures with two fences parallel to one side as shown below. What is the shortest length of fence that the rancher can use?



Solution: About 19.6 miles of fencing is needed. See the diagram below. Note that the total amount of fencing needed, based on the labeling of the figure is $2y + 4x$ and the area fenced in is $A = 12 = xy$. Solve the last relation for y to get $y = 12/x$. Now the amount of fencing f can be written in terms of x as follows: $f(x) = 2(12/x) + 4x, 0 < x$. Find the critical points of f by first noting that $f'(x) = 4 - 24x^{-2}$. Then solve $f'(x) = 0$ to get $x = \sqrt{6}$. The sign chart for f' shows that f has a minimum at $\sqrt{6}$. The rancher needs $f(\sqrt{6}) = 8\sqrt{6} \approx 19.6$ miles of fencing.



14. (10 points) The line $2x + 3y = 13$ is tangent to the graph of a function $g(x)$ at the point $(2, 3)$. What is $g'(2)$?

Solution: The slope of the line is $m = -2/3$, so $g'(2) = -2/3$.

15. (10 points) There is exactly one function $f(x)$ such that $f'(x) = e^x - x^e$ and $f(0) = 2$. Find the value of this function at $x = 1$. Leave your answer as a decimal accurate to the nearest tenth.

    Solution: Antidifferentiating, we have $f(x) = e^x - x^{e+1}/(e+1) + C$, and since $f(0) = 2$, it follows that $C = 1$. So $f(1) = e - \frac{1}{e+1} + 1 \approx 3.45$.

16. (70 points) Evaluate each of the following integrals using the Fundamental Theorem of Calculus (ie, antidifferentiate, then measure the growth of an antiderivative over the interval).

   (a) Evaluate $\displaystyle\int_0^e \frac{2x}{x^2+1}\,dx$

   Solution: $\int_0^e \frac{2x}{x^2+1}\,dx = \ln(x^2+1)|_0^e = \ln(e^2+1) - \ln(1) = \ln(e^2+1) \approx 2.127$.

   (b) Evaluate $\displaystyle\int_{-1}^{1} (x+2)^5 x\,dx$. Leave your answer as a decimal accurate to the nearest hundredth.

   Solution: Let $u = x+2$. The $du = dx$ and $\displaystyle\int (x+2)^5 x\,dx = \int u^5(u-2)\,du = \int u^6 u - 2u^5\,du = u^7/7 - 2u^6/6 = (x-2)^7/7 - (x-2)^6/3$. So $\displaystyle\int_{-1}^{1} (x+2)^5 x\,dx = \left(\frac{3^7}{7} - \frac{3^6}{3}\right) - \left(\frac{1}{7} - \frac{1}{3}\right) \approx 69.62$.

   (c) Evaluate $\displaystyle\int_0^2 x^2 - \sqrt{x}\,dx$

   Solution: $\int_0^2 x^2 - \sqrt{x}\,dx = \int_0^2 x^2\,dx - \int_0^2 \sqrt{x}\,dx = \frac{x^3}{3} - \frac{2x^{3/2}}{3}|_0^2 = \frac{8-4\sqrt{2}}{3} \approx 0.781$.

   (d) Evaluate $\displaystyle\int_{-1}^{1} \frac{d}{dx}\,(5x^4 - 3x^3 + 7x)\,dx$.

   Solution: An antiderivative of $\frac{d}{dx}\,(5x^4 - 3x^3 + 7x)$ is $5x^4 - 3x^3 + 7x$, which grows from 1 to 9 over the interval $[-1, 1]$. So the integral is 8.

(e) Evaluate $\displaystyle\int_0^4 \frac{x^3+8}{x+2}\,dx$

Solution: Factor the numerator to get $\displaystyle\int_0^4 \frac{x^3+8}{x+2}\,dx = \int_0^4 \frac{(x+2)(x^2-2x+4)}{x+2} = \int_0^4 x^2-2x+4 = x^3/3 - x^2 + 4x|_0^4 = 64/3 - 16 + 16 = 64/3.$

(f) Evaluate $\displaystyle\int_1^3 x^3\cdot(x^4-2)^2\,dx$

Solution: Let $u = x^4 - 2$. Then $du = 4x^3 dx$ and $\int_1^3 x^3\cdot(x^4-2)^2\,dx = \frac{1}{4}(x^4-2)^3|_1^3 = \frac{79^3}{12} - \frac{-1}{12} = 41086.5.$

(g) Evaluate $\displaystyle\int_0^4 2e^{2x}\,dx$

Solution: $\int_0^4 2e^{2x}\,dx = e^{2x}|_0^4 = e^8 - e^0 \approx 2980.95 - 1 = 2979.95.$