May 8, 2013 **Name** ______________________________

The total number of points available is 297. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit.

1. (30 points) Limit Problems. Compute each of the following limits:

   (a) $\lim_{x\to 1} \dfrac{x^2+3x-4}{x^2-4x+3}$

   Solution: We can rewrite the problem after factoring as $\lim_{x\to 1} \dfrac{(x+4)(x-1)}{x-3)(x-1)}$, which goes to $-5/2$ as x goes to 1.

   (b) $\lim_{x\to 2} \dfrac{\sqrt{x^2-3}-1}{x-2}$

   Solution: We can eliminate the radical by rationalizing. $\frac{\sqrt{x^2-3}-1}{x-2} = \frac{\sqrt{x^2-3}-1}{x-2} \cdot \frac{\sqrt{x^2-3}+1}{\sqrt{x^2-3}+1} = \frac{x^2-3-1}{(x-2)\cdot(\sqrt{x^2-3}+1)} = \frac{x+2}{\sqrt{x^2-3}+1}$ which goes to 2 as x goes to 2.

   (c) $\lim_{x\to 3} \dfrac{\frac{1}{x-2}-1}{x-3}$

   Solution: Do the fractional arithmetic to get $\lim_{x\to 3} \dfrac{3-x}{x-3} \cdot \dfrac{1}{x-2} = -1$

2. (20 points) Derivative Problem.
   Let $f(x) = \sqrt{2x+1}$. Then $f'(x) = 1/\sqrt{2x+1}$. Recall that $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$.
   Use this limit definition of derivative to verify that $f'(x) = 1/\sqrt{2x+1}$.

   Solution:

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} &= \lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{2(x+h)+1}-\sqrt{2x+1}}{h}\cdot\frac{\sqrt{2(x+h)+1}+\sqrt{2x+1}}{\sqrt{2(x+h)+1}+\sqrt{2x+1}} \\
&= \lim_{h\to 0}\frac{2(x+h)+1-(2x+1)}{h\cdot(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \\
&= \lim_{h\to 0}\frac{2h}{h\cdot(\sqrt{2(x+h)+1}+\sqrt{2x+1})} \\
&= \lim_{h\to 0}\frac{2}{\sqrt{2(x+h)+1}+\sqrt{2x+1}} \\
&= 1/\sqrt{2x+1}
\end{aligned}
$$

3. (15 points) Consider the function $f(x) = (x + x^2 - e^{2x})^2$.

   (a) Compute $f'(x)$
   Solution: Since $f'(x) = 2(x + x^2 - e^{2x})^1 \cdot (1 + 2x - 2e^{2x})$.

   (b) Find an equation of the line tangent to the graph of f at the point $(0, f(0))$.
   Solution: $f(0) = (-1)^2 = 1$, and $f'(0) = 2(1)(1-2) = -2$ so the line is $y - 1 = -2(x - 0)$, or $y = -2x + 1$.

4. (15 points) Find an interval over which the function

$$G(x) = \ln(x^3 + x^2 + 1), \qquad -1 \le x,$$

is decreasing.

Solution: First, the derivative of G is $G'(x) = \frac{3x^2+2x}{x^3+x^2+1}$. We just need to find out when $G'(x)$ is negative. The denominator is positive for all $x \ge -1$, and the numerator is negative between $-2/3$ and 0. So G is decreasing on the interval $(-2/3, 0)$.

5. (15 points) The function $h(x) = (3x-2)^2 \cdot x^5$ has three critical points, $x = 2/3$, $x = 0$ and a third point.

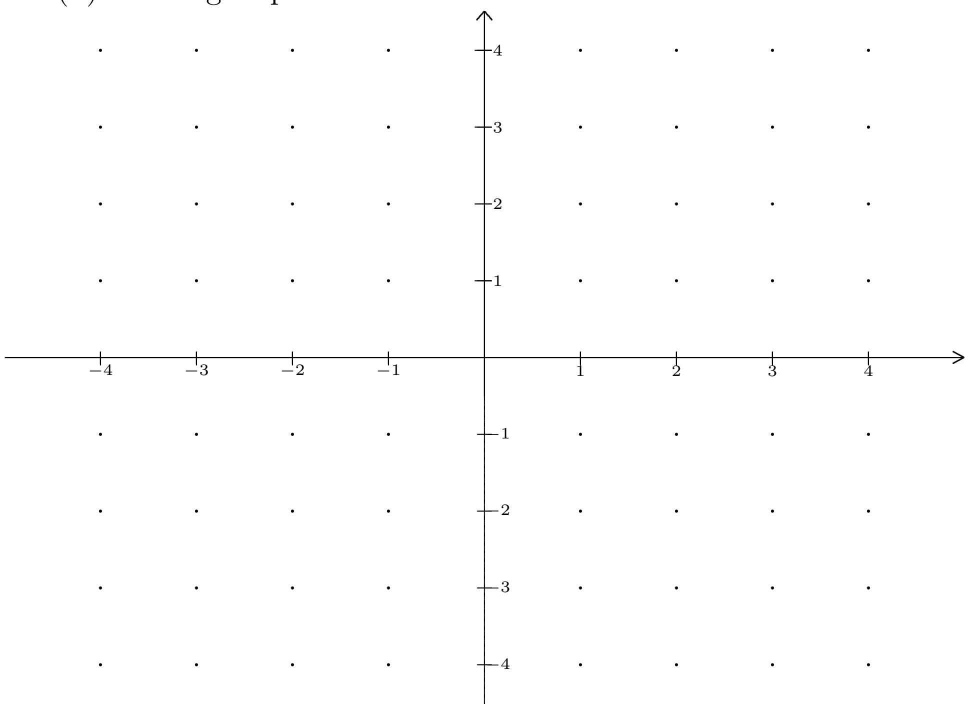
   (a) Find the third critical point.

   Solution: $h'(x) = 3{\cdot}2{\cdot}(3x-2)x^5+5x^4(3x-2)^2 = (3x-2)x^4[6x+5(3x-2)]$, which has a zero at $x = 10/21$.

   (b) At which of the critical points does h have a local maximum, a local minimum, or neither? In other words describe the nature of each critical point.

   Solution: Looking at the sign chart for h', we see that h has a local minimum at 2/3, a local maximum at 10/21 and neither at 0.

6. (30 points) There is a rational function $r(x)$ with exactly three zeros at $x = -2, x = 2$, and $x = 4$ and two vertical asymptotes, $x = 0$ and $x = 5$. Also, $r(x)$ has a horizontal asymptote $y = 2$. Find a symbolic representation of $r(x)$ and build the sign chart for it. The symbolic representation is not unique. Does your r have a relative max or min near 3? If so, which one. Sketch the graph of $r(x)$ on the grid provided.



Solution: One such r is given by $r(x) = \frac{2(x^2-4)(x-4)}{x^2(x-5)}$. You must be sure that the numerator and denominator have the same degree. That minimum degree is 3. Now this function is positive on each of the intervals $(-\infty, -2), (2, 4)$, and $(5, \infty)$. So my r has a relative maximum near $x = 3$.

7. (40 points) Let $f(x) = \dfrac{x}{2} + 1$ The region bounded by f over the interval $[0, 4]$. is a trapezoid T. Specifically, $T = \{(x, y) \mid 0 \leq x \leq 4, 0 \leq y \leq f(x)$.

   (a) Use geometry to find the area of T.

   Solution: The region T can be broken into a rectangle of area 4 and a right triangle of area 4, so the area of T is 8.

   (b) Build the Riemann sum for f over $[0, 4]$ using $n = 4$ subintervals of equal length and using the right endpoints as the sample points to determine the height of each rectangle. Is the approximation an over-estimate or an under-estimate?

   Solution: The sum is $f(1)(1-0)+f(2)(2-1)+f(3)(3-2)+f(4)(4-3) = 3/2 + 2 + 5/2 + 3 = 9$.

   (c) Use calculus to find the area of the region T. Your calculation must show what antiderivative you used and how you measured its growth.

   Solution: $\int_0^4 \frac{x}{2} + 1\ dx = \frac{x^2}{4} + x|_0^4 = \frac{16}{4} + 4 - 0 = 8$.

8. (42 points) Find the following antiderivatives.

   (a) $\int 3x - 5\ dx$

   Solution: $x^2 - 5x + C$.

   (b) $\int 9x^3 - 4x - 2x^{-1}\ dx$

   Solution: $\int 9x^3 - 4x - 2x^{-1}\ dx = 9 \cdot \dfrac{1}{4}x^4 - 2 \cdot x^2 - 2\ln x + C$.

   (c) $\int \dfrac{3x^3 + 2x^2 - 1}{x^2}\ dx$

   Solution: $\int (3x^3 + 2x^2 - 1)/x^2\ dx = \int 3x + 2 - x^{-2} dx = 3x^2/2 + 2x + x^{-1} + C$.

   (d) $\int \dfrac{2x}{x^2 + 3}\ dx$

   Solution: By substitution, $(u = x^2 + 3)$, $\int \frac{2x}{x^2+3}\ dx = \ln|x^2 + 3| + C$.

   (e) $\int 4x^3\sqrt{x^4 + 3}\ dx$

   Solution: By substitution with $u = x^4 + 3$, $\int 4x^3\sqrt{x^4 + 3}\ dx = \frac{2}{3}(x^4 + 3)^{3/2} + C$.

   (f) $\int 3x^2 e^{x^3}\ dx$

   Solution: By substitution with $u = x^3, du = 3x^2$, $\int e^u\ du = e^u + C = (1/3)e^{x^3} + C$.

9. (15 points) The percentage of alcohol in a person's bloodstream t hours after drinking 4 fluid ounces of whiskey is given by

$$A(t) = 0.24te^{-0.3t}, \qquad 0 \le t \le 6.$$

   (a) How fast is the percentage of alcohol in the person's bloodstream changing after 1 hour?

   Solution: First, find the derivative of A using the product rule. $A'(t) = 0.24e^{-0.3t} - 0.3 \cdot 0.24te{-}0.3t$, so we find that $A'(1) = e^{-0.3}(0.24 - 0.072) \approx 0.1244$, which mean that the percentage is growing at about 12% per hour.

   (b) At what time is the percentage maximized?

   Solution: Find the critical points of A by setting $A'(t)$ equal to zero. $e^{-.3t}(.3(.24)t - .24) = 0$ has just one solution, $t \approx 10/3$ hours.

   (c) What is that maximum percentage?

   Solution: Evaluating A at 10/3 gives $A(10/3) \approx .2943$.

10. (15 points) Let $G(x) = \sqrt{x^2(2x-5)(3x+7)}$. Note that $G(3) = \sqrt{9 \cdot 1 \cdot 16} = 12$, so G is defined at 3. Find the domain of $G(x)$. Express your answer in interval notation.

    Solution: We can instead solve the inequality $f(x) = x^2(2x-5)(3x+7) \ge 0$. Using the Test Interval technique we see that $f(x)$ is nonnegative precisely on $(-\infty, -7/3] \cup [5/2, \infty)$.

11. (15 points) There is one point on the graph of the function $f(x) = \ln(x^2 + x)$ where the line tangent to the graph has a slope of 3. Find the x-coordinate of that point.

    Solution: Since $f'(x) = \frac{2x+1}{x^2+x}$, we can solve $\frac{2x+1}{x^2+x} = 3$ using the quadratic formula. We get two numbers, $x = \frac{-1\pm\sqrt{13}}{6}$, but one of these is not in the domain of f, so the point $x = \frac{-1+\sqrt{13}}{6} \approx 0.43$ is the only one that works.

12. (15 points) Compute the number $\int_0^4 (x-2)^4 \cdot (x+2)\ dx$.

    Solution: Let $u = x - 2$, Then $du = dx$ and $x + 2 = u + 4$. The integral turns into $\int u^4(u+4)\ du = u^6/6 + 4u^5/5$. Replacing u with $x - 2$, we get $(x-2)^6/6 + 4(x-2)^5/5|_0^4 = 4 \cdot 2^6/5 = 256/5 = 51.2$.

13. (30 points) A function f satisfies $f(3) = 2$. The line tangent to the graph of f at $(3, 2)$ is given by $y = 2x - 4$.

    (a) What is $f'(3)$?

    Solution: $f'(3)$ is the slope of the tangent line, 2.

    (b) Suppose that $f''(x) = x - 4$. What is $f'(2)$?

    Solution: First note that $f'(x) = x^2/2 - 4x + C$ for some constant C. Solve $f'(3) = 2$ for this to get $C = 19/2$. So $f'(2) = 2^2/2 - 4 \cdot 2 + 19/2 = 7/2$.

    (c) Find a representation for f.

    Solution: $f(x) = x^3/6 - 2x^2 + 19x/2 + C$ for some constant C. We can solve for C since we know that $f(3) = 2$. We get $C = -13$, so $f(x) = x^3/6 - 2x^2 + 19x/2 - 13$.