February 12, 2014 **Name** ______________________________

The problems count as marked. The total number of points available is 174. Throughout this test, **show your work.**

1. (10 points) A line L is given by the equation $2x + 3y = 6$. Another line L' perpendicular to L passes through the point $(2, 5)$. Find the y-intercept of L'. Then find the x-intercept of L'.

   Solution: The slope of L is $-2/3$, so the slope of L' is $3/2$. An equation for L' is therefore $y - 5 = (3/2)(x - 2)$. When $x = 0$, we get $y = 2$. On the other hand, the x-intercept is the value of x when $y = 0$, which is $x = -4/3$.

2. (10 points) Find all solutions to $||3x - 5| - 3| = 4$.

   Solution: First note that $|3x - 5| - 3$ could be either 4 or -4. That is, $|3x-5|-3 = 4$ or $|3x-5|-3 = -4$. Thus either $|3x-5| = 7$ or $|3x-5| = -1$. The first can be solved, the second cannot. The solutions are $x = 4$ and $x = -2/3$.

3. (10 points) Find the exact value of the expression

   $$|3\pi - 8| + |2\pi - 4| + |5\pi - 17|.$$

   Use the symbol π in your answer if you need to.

   Solution: Using the definition of absolute value, we have $3\pi - 8 + 2\pi - 4 - (5\pi - 17) = -8 - 4 + 17 = 5$.

4. (10 points) What is the distance from the center of the circle $x^2 + y^2 + 4y = 21$ to the point $(3, 2)$? Is the point $(3, 2)$ **inside**, **outside**, or **on** the the circle?

   Solution: Complete the square to find the center and radius. $x^2+y^2+4y+4 = 25$, so $(x - 0)^2 + (y + 2)^2 = 5^2$ and we see that the circle has center $(0, -2)$ and radius $r = 5$. The distance from $(0, -2)$ to $(3, 2)$ is $\sqrt{3^2 + 4^2} = 5$, so the point is **on** the circle.

5. (30 points) Evaluate each of the limits indicated below.

   (a) $\displaystyle\lim_{x\to-\infty} \frac{3x^4-6}{(11-3x^2)^3}$

   Solution: The degree of the numerator is 4 while the degree of the denominator is 6, so the limit is 0.

   (b) $\displaystyle\lim_{x\to1} \frac{(x+1)^2-4}{(x+2)^2-9}$

   Solution: Factor both numerator and denominator to get $\displaystyle\lim_{x\to1} \frac{(x-1)(x+3)}{(x-1)(x+5)} =$ $\displaystyle\lim_{x\to1} \frac{(x+3)}{(x+5)} = 2/3$

   (c) $\displaystyle\lim_{h\to0} \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$.

   Solution: Do the fractional arithmetic to get $\displaystyle\lim_{h\to0} \frac{x-(x+h)}{h(x+h)(x)}$ which is found to be $-\frac{1}{x^2}$.

   (d) $\displaystyle\lim_{x\to\infty} \frac{(2x^2+3)^3}{(3x^3+x-2)^2}$

   Solution: The numerator has the form $8x^6+\cdots$ while the denominator has the form $9x^6+\cdots$ so the limit is 8/9.

   (e) $\displaystyle\lim_{h\to0} \frac{\sqrt{25+2h}-5}{h}$

   Solution: Rationalize the numerator to get $\displaystyle\lim_{h\to0} \frac{\sqrt{25+2h}-5}{h}$ $= \lim_{h\to0} \frac{(\sqrt{25+2h}-5)(\sqrt{25+2h}+5)}{h(\sqrt{25+2h}+5)}$. This reduces to $\displaystyle\lim_{h\to0} \frac{(\sqrt{25+2h}-5)(\sqrt{25+2h}+5)}{h(\sqrt{25+2h}+5)} =$ $\displaystyle\frac{25+2h-25}{h(\sqrt{25+2h}+5)} = \lim_{h\to0} \frac{2}{\sqrt{25+2h}+5} = \frac{2}{2\sqrt{25}} = \frac{1}{5}$.

   (f) $\displaystyle\lim_{x\to3} \frac{x^3-27}{x-3}$

   Solution: Factor x^3-27 into $(x-3)(x^2+3x+9)$ and remove the $x-3$'s from the fraction. Then the limit is $3^2+3\cdot3+9=27$.

6. (12 points) The points $(1, 0)$, $(5, 1)$, (u, v), and $(0, 4)$ are the vertices of a square. Find u and v.

   Solution: One way to think about this is line segment through $(5, 1)$ and (u, v) must have the same slope and length as the segment from $(1, 0)$ to $(0, 4)$. But an easier way to think about this is that $(0, 4)$ is up 4 and over 1 from $(1, 0)$, so (u, v) must be up 4 and over 1 from $(5, 1)$, That is $(u, v) = (5-1, 1+4) = (4, 5)$.

7. (12 points) Find the domain of the function

$$g(x) = \frac{\sqrt{x^2 - 2x - 3}}{x - 9}.$$

   Express your answer as a union of intervals. That is, use interval notation.

   Solution: The numerator is defined for $(x+1)(x-3) \geq 0$, that is $(-\infty, -1) \cup (3, \infty)$. The denominator is zero at $x = 9$, so this number must be removed. We need to include the numbers $x = -1$ and $x = 3$ and pluck out the 9. Thus, the domain is $(-\infty, -1] \cup [3, 9) \cup (9, \infty)$.

8. (12 points) Let $H(x) = (x^2-4)^2(x-3)^3$. Using the chain rule and the product rule,

$$H'(x) = 2(x^2 - 4) \cdot 2x(x - 3)^3 + (x^2 - 4)^2 \cdot 3(x - 3)^2.$$

   Find all five zeros of $H'(x)$.

   Solution: Factor out the common terms to get $H'(x) = (x^2-4)(x-3)^2[3(x^2-4)+4x(x-3)]$. One factor is $4x(x-3)+3(x^2-4) = 7x^2-12x-12$. Apply the quadratic formula to get $x = \frac{12\pm\sqrt{144+12\cdot 28)}}{14}$ which reduces to $x = \frac{12\pm 4\sqrt{30}}{14} = \frac{6\pm 2\sqrt{30}}{7}$. The other three zeros are $x = 3$ and $x = \pm 2$.

9. (21 points) Let

$$f(x) = \begin{cases} 2x+3 & \text{if } -1 < x \leq 0 \\ |x-3| & \text{if } 0 < x < 4 \\ 2 & \text{if } x = 4 \\ 5-x & \text{if } 4 < x \leq 6 \end{cases},$$

(a) What is the domain of f? Express your answer in interval notation.
Solution: $(-1,0] \cup (0,4) \cup \{4\} \cup (4,6] = (-1,6]$.

(b) What is $\lim_{x \to 0^-} f(x)$?
Solution: $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} 2x+3 = 3$.

(c) What is $\lim_{x \to 0^+} f(x)$?
Solution: $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} |x-3| = 3$.

(d) Is f continuous at $x = 0$? Discuss why or why not.
Solution: f is continuous at $x = 0$ because $f(0) = \lim_{x\to 0} f(x) = 3$.

(e) What is $\lim_{x \to 4^-} f(x)$?
Solution: $\lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} |x-3| = 1$.

(f) What is $\lim_{x \to 4^+} f(x)$?
Solution: $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} |x-3| = 1$.

(g) Is f continuous at $x = 4$? Discuss why or why not.
Solution: f is not continuous at $x = 4$ because $f(4) = 2 \neq \lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2} 5 - x = 1$.

10. (20 points) Let $f(x) = \sqrt{3x-2}$.

(a) Let h be a positive number. What is the slope of the line passing through the points $(6, f(6))$ and $(6+h, f(6+h))$. Your answer depends on h, of course. Suppose your answer is called $G(h)$.

Solution: $\frac{\sqrt{3(6+h)-2}-4}{h}$, since $f(6) = 4$.

(b) Compute $\lim_{h\to 0} G(h)$.

Solution: Since we get zero over zero, we recall that, in this case, we should rationalize the denominator.

$$\begin{aligned}
\lim_{h\to 0}\frac{f(6+h)-f(6)}{h} &= \lim_{h\to 0}\frac{\sqrt{3(6+h)-2}-\sqrt{36-2}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{3(6+h)-2}-\sqrt{36-2}}{h}\cdot\frac{\sqrt{3(6+h)-2}+\sqrt{36-2}}{\sqrt{3(6+h)-2}+\sqrt{36-2}} \\
&= \lim_{h\to 0}\frac{3(6+h)-2-(36-2)}{h(\sqrt{3(6+h)-2}+\sqrt{36-2})} \\
&= \lim_{h\to 0}\frac{3h}{h(\sqrt{3(6+h)-2}+\sqrt{36-2})} \\
&= \lim_{h\to 0}\frac{3}{(\sqrt{3(6+h)-2}+\sqrt{3\cdot 6-2})} \\
&= \frac{3}{2(\sqrt{3\cdot 6-2})} = \frac{3}{8}
\end{aligned}$$

(c) Your answer to (2) is the slope of the line tangent to the graph of f at the point $(6, f(6))$. In other words, your answer is $f'(6)$. Write and equation for the tangent line.

Solution: The line is $y - 4 = 3(x-6)/8$, or $y = 3x/8 + 7/4$.

11. (12 points) Let $f(x) = (2x-3)^5(5x^2-1)+17x^5$, let $g(x) = (x-4)^4(8x^3)-2x^4$.

    (a) What is the degree of the polynomial $f - g$?

    Solution: 7

    (b) What is the degree of the polynomial $f \cdot g$?

    Solution: 14

    (c) Estimate within one tenth of a unit the value of $f(10000)/g(10000)$.

    Solution: Any answer between 19.9 and 20.1 works. See the next part.

    (d) Compute $\lim_{x\to\infty} \frac{f(x)}{g(x)}$.

    Solution: $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{(2x)^5(5x^2)}{x^4 \cdot 8x^3} = \lim_{x\to\infty} \frac{160x^7}{8x^7} = 20$ because the degree of the denominator is the same as that of the numerator.

12. (15 points) Recall that the Intermediate Value Theorem guarantees that for any function f continuous over the interval $[a, b]$ and for any number M between $f(a)$ and $f(b)$, there exists a number c such that $f(c) = M$. The function $f(x) = \frac{1}{1+\frac{1}{x}}$ is continuous for all $x > 0$. Let $a = 1$.

    (a) Pick a number $b > 1$ (any choice is right), and then find a number M between $f(a)$ and $f(b)$.

    Solution: Suppose you picked $b = 2$. Then $f(a) = 1/2$ and $f(b) = 2/3$. You could choose $M = 3/5$.

    (b) Show that the conclusion to the Intermediate Value Theorem is satisfied by finding a number c in (a, b) such that $f(c) = M$.

    Solution: To solve $f(c) = 3/5$, write $\frac{1}{1+\frac{1}{x}} = 3/5$, from which we get $5 = 3 + 3/x$ and then $3/x = 2$, so $x = 3/2$. Indeed $3/2$ is between 1 and 2, as required.