October 2, 2014 **Name** ______________________________

The problems count as marked. The total number of points available is 153. Throughout this test, **show your work.**

1. (10 points) Find all solutions to $||2x - 15| - 3| = 2$.

   Solution: First note that $|2x - 15| - 3$ could be either 2 or -2. That is, $|2x - 15| - 3 = 2$ or $|2x - 15| - 3 = -2$. Thus either $|2x - 15| = 5$ or $|2x - 15| = 1$. Each of these has two solutions. The former, $x = 10$ and $x = 5$ and the later, $x = 8$ and $x = 7$.

2. (24 points) The set of points C_1 in the plane satisfying $x^2 + y^2 - 4y = 0$ is a circle. The set C_2 whose points satisfy $x^2 - 24x + y^2 - 14y = -49$ is also a circle.

   (a) What is the distance between the centers of the circles?

   Solution: The centers are $(0, 2)$ and $(12, 7)$, so the distance is $d = \sqrt{12^2 + (7-2)^2} = \sqrt{169} = 13$.

   (b) How many points in the plane belong to both circles. That is, how many points in the plane satisfy both equations?

   Solution: The radii are 2 and 12 and the centers are 13 units apart, so the circles have two points in common.

   (c) Find an equation for the line connecting the centers of the circles.

   Solution: The slope is $(7-2)/(12-0) = 5/12$. Using the point-slope form, we have $y - 2 = (5/12)(x - 0)$, or $y = 5x/12 + 2$.

3. (35 points) Evaluate each of the limits indicated below.

(a) $\lim_{x\to\infty} \dfrac{2x^6 - 6}{(11 - 2x^2)^3}$

Solution: The degrees of the numerator and the denominator are both 6, so the limit is $2/-8 = -1/4$.

(b) $\lim_{x\to 1} \dfrac{x^4 - 1}{x^2 - 1}$

Solution: Factor both numerator and denominator to get $\lim_{x\to 1} \frac{x^4-1}{x^2-1} = \lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{(x^2-1)} = \lim_{x\to 1} \frac{x^2+1}{1} = 2$

(c) $\lim_{h\to 0} \dfrac{(2+h)^3 - 8}{h}$.

Solution: Expand the numerator to get

$$\lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h\to 0} \frac{h(12 + 6h + h^2)}{h}$$

$= \lim_{h\to 0}(12+6h+h^2)$, and now the zero over zero problem has disappeared. So the limit is 12.

(d) $\lim_{x\to 1} \dfrac{x^2 - 4x + 3}{x^2 + x - 2}$

Solution: Factor and eliminate the $x - 1$ from numerator and denominator to get

$$\lim_{x\to 1} \frac{x-3}{x+2} = -2/3$$

(e) $\lim_{x\to 2} \dfrac{\frac{1}{3x} - \frac{1}{6}}{\frac{1}{2x} - \frac{1}{4}}$

Solution: The limit of both the numerator and the denominator is 0, so we must either factor or do the fractional arithmetic. Factoring seems to work best. The limit becomes

$$\lim_{x\to 2} \frac{\frac{1}{3}[\frac{1}{x} - \frac{1}{2}]}{\frac{1}{2}[\frac{1}{x} - \frac{1}{2}]} = \lim_{x\to 2} \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

(f) $\lim_{x\to -\infty} \dfrac{\sqrt{36x^2 - 3x}}{9x - 11}$

Solution: Divide both numerator and denominator by x to get $\lim_{x\to -\infty} \frac{-\sqrt{36-3/x}}{9-11/x} = 6/9 = -2/3$ because the degree of the denominator is essentially the same as that of the numerator.

(g) $\lim_{x\to 2} \frac{\sqrt{8x}-4}{x-2}$

Solution: Rationalize the numerator to get

$$\lim_{x\to 2} \frac{8x-16}{x-2}\frac{1}{\sqrt{8x}+4} = 8\cdot\frac{1}{8} = 1$$

4. (12 points) Find the domain of the function

$$g(x) = \frac{\sqrt{(x+10)(2x-3)(3x-17)}}{x^2-4}.$$

Express your answer as a union of intervals. That is, use interval notation.

Solution: Using the test interval technique, we see that the numerator is defined for when x belongs to $[-10, 3/2) \cup (17/3, \infty)$. The denominator is zero at $x = -2$ and $x = 2$, so these two numbers must be removed. Thus, the domain is $[-10, -2) \cup (-2, 3/2) \cup (17/3, \infty)$.

5. (12 points) Let $H(x) = (x^2 - 4)^2(2x + 3)^3$. Using the chain rule and the product rule,

$$H'(x) = 2(x^2 - 4)\cdot 2x(2x + 3)^3 + (x^2 - 4)^2 \cdot 3(2x + 3)^2 \cdot 2.$$

Three of the zeros of $H'(x)$ are $x = \pm 2$ and $x = -3/2$. Find the other two.

Solution: Factor out the common terms to get $H'(x) = (x^2-4)(2x+3)^2[4x+6(x^2-4)]$. One factor is $2x(2x + 3) + 3(x^2 - 4) = 7x^2 + 6x - 12$. Apply the quadratic formula to get $x = \frac{-7\pm\sqrt{36-4\cdot 6(-12)}}{14}$ which reduces to $x = \frac{-7\pm 2\sqrt{93}}{14}$.

6. (25 points) Given two functions,

$$g(x) = 2x + 1$$

and

$$f(x) = \begin{cases} x^2 & \text{if } x \leq 2 \\ -2 & \text{if } x > 3 \end{cases}$$

Use 'dne' for 'does not exist.'

(a) Write the domain of f in interval notation.
Solution: $(-\infty, 2) \cup (3, \infty)$.

(b) Compute $\lim_{x \to 3^+} f(x)$
Solution: -1

(c) Compute $\lim_{x \to 3^-} f(x)$
Solution: dne

(d) Complete the following table.

x	$g \circ f(x)$
-2	
-1	
0	
1	
2	
3	
π	

Solution:

x	$g \circ f(x)$
-2	9
-1	3
0	1
1	3
2	9
3	dne
π	-3

(e) Find the symbolic representation of $g \circ f(x)$
Solution:

$$g \circ f(x) = \begin{cases} 2x^2 + 1 & \text{if } x \leq 2 \\ -3 & \text{if } x > 3 \end{cases}$$

7. (25 points) Let $f(x) = \sqrt{4x-3}$.

   (a) Let h be a positive number. What is the slope of the line passing through the points $(3, f(3))$ and $(3+h, f(3+h))$. Your answer depends on h, of course. Suppose your answer is called $G(h)$.

   Solution: Letting $(x_1, y_1) = (3, f(3)$ and $(x_2, y_2) = (3+h, f(3+h)$, we have $\frac{f(3+h)-f(3)}{3+h-3} = \frac{\sqrt{4(3+h)-3}-3}{h}$.

   (b) Compute $\lim_{h\to 0} G(h)$.

   Solution: Since we get zero over zero, we recall that, in this case, we should rationalize the numerator.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} &= \lim_{h\to 0} \frac{\sqrt{4(3+h)-3}-\sqrt{12-3}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{4(3+h)-3}-3}{h} \cdot \frac{\sqrt{4(3+h)-3}+3}{\sqrt{4(3+h)-3}+3} \\
&= \lim_{h\to 0} \frac{4(3+h)-3-9}{h(\sqrt{4(3+h)-3}+3)} \\
&= \lim_{h\to 0} \frac{4h}{h\left(\sqrt{4(3+h)-3}+3\right)} \\
&= \lim_{h\to 0} \frac{4}{\sqrt{4(3+h)-3}+3} \\
&= \frac{4}{2(3)} = \frac{2}{3}
\end{aligned}$$

   (c) Your answer to (b) is the slope of the line tangent to the graph of f at the point $(3, f(3))$. In other words, your answer is $f'(3)$. Write and equation for the tangent line.

   Solution: The line is $y - 3 = 2(x-3)/3$, or $y = 2x/3 + 1$.

8. (10 points) Evaluate the following limits.

   (a) $\lim_{x\to\infty} \frac{(2-x)(10+6x)}{(3-5x)(8+8x)}$

   Solution: The coefficient of the x^2 term in the numerator is -6 and the coefficient of the x^2 term in the denominator is -40, so the limit is $-6/-40 = 3/20$.

(b) $\lim_{x\to-\infty} \frac{(2-x)(10+6x)}{(3-5x)(8+8x)}$

Solution: The coefficient of the x^2 term in the numerator is -6 and the coefficient of the x^2 term in the denominator is -40, so the limit is $-6/-40 = 3/20$.