October 14, 2015 **Name** ____________________________________

The problems count as marked. The total number of points available is 171. Throughout this test, **show your work.** Use of calculator to circumvent ideas discussed in class will generally result in no credit.

1. (20 points)

(a) Find all solutions to $||x-3|-8|=5$.

(b) Find all solutions: $|x-3|+|x-8|=8$.

Solution: First note that $|x-3|-8$ could be either 5 or -5. That is, $|x-3|-8=5$ or $|x-3|-8=-5$. Thus either $|x-3|=13$ or $|x-3|=3$. Each of these has two solutions. The former, $x=-10$ and $x=16$ and the later, $x=6$ and $x=0$.

(c) Find the domain of the function $f(x)=\sqrt{||x-3|-8|-5}$ and write your answer in interval form.

Solution: Solve the inequality $||x-3|-8|-5\geq 0$. Change it to an equality and solve to get the four numbers $0, 6, -10$, and 16. Then use the test interval technique to get $(-\infty,-10]\cup[0,6]\cup[16,\infty)$.

2. (24 points) The set of points C_1 in the plane satisfying $x^2+y^2=4$ is a circle. The set C_2 whose points satisfy $x^2-16x+y^2-12y=-36$ is also a circle.

(a) What is the distance between the centers of the circles?

Solution: The centers are $(0,0)$ and $(8,6)$, so the distance is $d=\sqrt{8^2+6^2}=\sqrt{100}=10$.

(b) How many points in the plane belong to both circles. That is, how many points in the plane satisfy both equations?

Solution: The radii are 2 and 8 ($x^2-16x+y^2-12y=-36$ so $x^2-16x+64+y^2-12y+36=-36+36+64=64$) and the centers are 10 units apart, so the circles have one point in common.

(c) Find an equation for the line connecting the centers of the circles.

Solution: The slope is $(6-0)/(8-0)=3/4$. Using the point-slope form, we have $y-0=(3/4)(x-0)$, or $y=3x/4$.

3. (35 points) Evaluate each of the limits indicated below.

(a) $\lim_{x \to 3} \dfrac{x^2 - 6x + 9}{x^2 + x - 12}$

Solution: Factor both the numerator and the denominator so that you can remove the common factor $x - 3$ from both. Then $\lim_{x\to 3} \frac{(x-3)(x-3)}{(x-3)(x+4)} = 0$.

(b) $\lim_{x \to 2} \dfrac{x^3 - 8}{x^3 - 4x^2 + 7x - 6}$

Solution: Factor both the numerator and the denominator so that you can remove the common factor $x - 2$ from both. Then $\lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{(x-2)(x^2-2x+3)} = \frac{12}{3} = 4$.

(c) $\lim_{x \to 4} \dfrac{x - 4}{x^3 - 64}$

Solution: Factor the denominator to get $\lim_{x\to 4} \frac{x-4}{(x-4)(x^2+4x+16)} = \lim_{x\to 4} \frac{1}{(x^2+4x+16)} = 1/48$

(d) $\lim_{x \to -3} \dfrac{x^3 + 27}{x + 3}$

Solution: Factor the numerator to get $\lim_{x\to -3} \frac{(x+3)(x^2-3x+9)}{x+3} = \lim_{x\to -3}(x^2 - 3x + 9) = 27$.

(e) $\lim_{x \to 2} \dfrac{\frac{1}{3x} - \frac{1}{6}}{x - 2}$

Solution: The limit of both the numerator and the denominator is 0, so we must do the fractional arithmetic. So we have $\lim_{x \to 2} \dfrac{\frac{1}{3x} - \frac{1}{6}}{x - 2} = \lim_{x \to 2} \dfrac{\frac{2}{6x} - \frac{x}{6x}}{x - 2} = \lim_{x \to 2} -\dfrac{x - 2}{(6x)(x - 2)} = -1/12$.

(f) $\lim_{x \to 5} \dfrac{\sqrt{4x + 5} - 5}{x - 5}$

Solution: Rationalize the numerator to get $\lim_{x \to 5} \dfrac{(\sqrt{4x + 5} - 5)(\sqrt{4x + 5} + 5)}{(x - 5)(\sqrt{4x + 5} + 5)} = \lim_{x \to 5} \dfrac{4x - 20}{(x - 5)(\sqrt{4x + 5} + 5)} = 4/10 = 2/5$.

(g) $\lim_{x \to \sqrt{8}} \dfrac{x^4 - 64}{x^2 - 8}$

Solution: Factor the numerator and remove the common factor to get $\lim_{x \to \sqrt{8}} \dfrac{(x^2 - 8)(x^2 + 8)}{x^2 - 8} = 16$.

(h) $\lim_{x\to\infty} \dfrac{(2x-3)^2(3x+1)}{(6x-1)^3}$

Solution: The degrees are the same, both 3, so the limit is the same as $\lim_{x\to\infty} \dfrac{12x^3}{216x^3} = 1/18$.

4. (12 points) Find the domain of the function

$$g(x) = \frac{\sqrt{(x^2-16)(2x-3)}}{x^2-9}.$$

Express your answer as a union of intervals. That is, use interval notation.

Solution: Using the test interval technique, we see that the numerator is defined for when x belongs to $[-4, 3/2] \cup [4, \infty)$. The denominator is zero at $x = -3$ and $x = 3$, so these two numbers must be removed. But 3 does not belong anyway. Thus, the domain is $[-4, -3) \cup (-3, 3/2] \cup [4, \infty)$.

5. (12 points) Let $H(x) = (x^2-9)^2(2x-3)^2$. Using the chain rule and the product rule,

$$H'(x) = 2(x^2-9) \cdot 2x(2x-3)^2 + 2(x^2-9)^2 \cdot 2(2x-3).$$

Three of the zeros of $H'(x)$ are $x = \pm 3$ and $x = 3/2$. Find the other two.

Solution: Factor out the common terms to get $H'(x) = 4(x^2-9)(2x-3)[x(2x-3)+(x^2-9)]$. One factor is $3x^2-3x-9$. Factor out the 3 and then apply the quadratic formula to get $x = \frac{1\pm\sqrt{4-4\cdot 3}}{2}$ which reduces to $x = \frac{1\pm\sqrt{13}}{2}$.

6. (15 points) Let $f(x) = (x^2 - 4)^4$

   (a) Find $f'(x)$

   Solution: Using the chain rule, we have $f'(x) = 4(x^2 - 4)^3 \cdot 2x$.

   (b) Use the information you found in (a) to find an equation for the line tangent to f at the point $(3, 625)$.

   Solution: Since $f'(3) = 4(9 - 4)^3 \cdot 2 \cdot 3 = 3000$, the tangent line is given by $y - 625 = 3000(x - 3)$.

   (c) Find all the critical points of f.

   Solution: Setting $f'(x) = 0$, we find three critical points, $x = \pm 2$ and $x = 0$.

7. (18 points) If a ball is shot vertically upward from the roof of 128 foot building with a velocity of 256 ft/sec, its height after t seconds is $s(t) = 128+256t-16t^2$.

   (a) What is the height the ball at time $t = 1$?

   Solution: $s(1) = 368$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) What is the maximum height the ball reaches?

   Solution: Solve $s'(t) = 256 - 32t = 0$ to get $t = 8$ when the ball reaches its zenith. Thus, the max height is $s(8) = 128 + 256(8) - 16(8)^2 = 1152$.

   (d) After how many seconds is the ball exactly 374 feet above the ground?

   Solution: Use the quadratic formula to solve $128 + 256t - 16t^2 = 374$. You get $t = 16 \pm 8\sqrt{3}$.

   (e) How fast is the ball going the first time it reaches the height 374 feet?

   Solution: Evaluate $s(t)$ when $t = 16 - 8\sqrt{3}$ to get $256(\sqrt{3} - 1)$ feet per second.

   (f) How fast is the ball going the second time it reaches the height 374 feet?

   Solution: Evaluate $s(t)$ when $t = 16 + 8\sqrt{3}$ to get $-(256(\sqrt{3} - 1))$ feet per second . In other words the ball is going downward at the same rate it was moving upwards when first went through 374 feet.

8. (10 points) The demand curve for a new phone is given by $3p+2x = 18$ where p is the price in hundreds of dollars and x is the number demanded in millions. The supply curve is given by $x - p^2 + 4p = 3$. Find the point of equilibrium.

   Solution: Since $x = -3p/2+9$ and $x = p^2 - 4p+3$, we can solve $-3p/2+9 = p^2 - 4p + 3$ for p: $p^2 - 4p + 3p/2 + 3 - 9 = 0$, so $p^2 - 5p/2 - 6 = 0$, so $2p^2 - 5p - 12 = 0$, which can be solved by factoring. $(2p+3)(p-4) = 0$, which has $p = 4$ and therefore $x = 3$ as a solution.

9. (25 points) Let $f(x) = \sqrt{x^2 - 5}$.

   (a) Let h be a positive number. What is the slope of the line passing through the points $(3, f(3))$ and $(3 + h, f(3 + h))$. Your answer depends on h, of course. Suppose your answer is called $G(h)$.

   Solution: Letting $(x_1, y_1) = (3, f(3)$ and $(x_2, y_2) = (3 + h, f(3 + h)$, we have $\frac{f(3+h)-f(3)}{3+h-3} = \frac{\sqrt{(3+h)^2-5}-\sqrt{3^2-5}}{h}$.

   (b) Compute $\lim_{h\to 0} G(h)$.

   Solution: Since we get zero over zero, we recall that, in this case, we should rationalize the numerator.

$$\begin{aligned}
\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} &= \lim_{h\to 0} \frac{\sqrt{4(3+h)^2-5}-\sqrt{9-5}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{(3+h)^2-5}-2}{h} \cdot \frac{\sqrt{(3+h)^2-5}+2}{\sqrt{(3+h)^2-5}+2} \\
&= \lim_{h\to 0} \frac{(3+h)-5-4}{h(\sqrt{(3+h)^2-5}+2)} \\
&= \lim_{h\to 0} \frac{9+6h+h^2-9}{h\left(\sqrt{(3+h)^2-5}+2\right)} \\
&= \lim_{h\to 0} \frac{6h+h^2}{h\sqrt{(3+h)^2-5}+2} \\
&= \frac{6}{4} = \frac{3}{2}
\end{aligned}$$

   (c) What is $f'(3)$?

   Solution: $f'(3) = 3/2$

   (d) Write an equation for the tangent line at $x = 3$.

   Solution: The line is $y - 2 = 3(x - 3)/2$.