September 18, 2001 Your name ____________________

The first 6 problems count 4 points each and the final ones counts as marked. Problems 1 through 6 are multiple choice. In the multiple choice section, circle the correct choice (or choices). You do not need to show your work on problems 1 through 6, but you must show your work on the other problems. The total number of points available is 125.

1. Which of the following is a factor of $x^4 - x$? Circle all those that apply.

   (A) x **(B)** $x-1$ **(C)** $x+1$ **(D)** x^2+x+1 **(E)** x^2-x+1

   Solution: Note that $x^4 - x = x(x^3-1) = x(x-1)(x^2+x+1)$, so the three answers are A,B, and D.

2. How many roots does the equation below have?

   $$x(x^2-3) - 4(x^2-3) = 0$$

   (A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

   Solution: D. Factor to get $x(x^2-3) - 4(x^2-3) = (x^2-3)(x-4) = 0$. Now the first factor has two zeros and the second has one, so there are 3 roots.

3. $$\frac{1+\frac{1}{x}}{1-\frac{1}{x}} =$$

   (A) $\dfrac{x+1}{x-1}$ **(B)** $\dfrac{x-1}{x+1}$ **(C)** $x-1$ **(D)** $1-x$ **(E)** x

   Solution: A. Note that $\frac{1+\frac{1}{x}}{1-\frac{1}{x}} = \frac{\frac{x+1}{x}}{\frac{x-1}{x}} = \frac{x+1}{x-1}$.

4. What is the radius of the circle whose equation is given by $x^2 - 8x + y^2 + 6y = 24$?

   (A) 4 **(B)** $\sqrt{24}$ **(C)** 5 **(D)** 6 **(E)** 7

   Solution: E. Complete the squares for each variable to get $x^2-8x+y^2+6y = x^2-8x+16+y^2+6y+9 = (x-4)^2+(y+3)^2 = 24+16+9 = 49 = 7^2$, so the center of the circle is $(4,-3)$ and the radius is $r=7$.

5. Which of the following is a solution to $2(5-3x)-2\cdot 5-3x=108$? Circle all that apply.

   (A) none **(B)** -12 **(C)** -9 **(D)** -2 **(E)** 0

   Solution: B. The equation is equivalent to $-9x=108$, or $x=-12$.

6. Which of the following is not a solution to $3(x-2)^3(x+1)^2-2(x-2)^2(x+1)^3=0$? Circle all that apply.

   (A) -2 **(B)** -1 **(C)** 0 **(D)** 2 **(E)** 8

   Solution: A and C. Factor to get $(x-2)^2(x+1)^2(3x-6-2x-2)=(x-2)^2(x+1)^2(x-8)=0$.

   On all the following questions, **show your work.**

7. (7 points) Find all roots of the equation

$$(x-1)(x+1)+(x-2)(x+1)=0.$$

   Solution: Factor $(x-1)(x+1)+(x-2)(x+1)$ to get $(x+1)((x-1)+(x-2))=(x+1)(2x-3)=0$, which has two roots, $x=-1$ and $x=3/2$.

8. (7 points) Rationalize the numerator of the expression $\frac{\sqrt{4+h}-2}{h}$, and express your answer in simplified form.

   Solution: $\frac{\sqrt{4+h}-2}{h}=\frac{\sqrt{4+h}-2}{h}\cdot\frac{\sqrt{4+h}+2}{\sqrt{4+h}+2}=\frac{4+h-4}{h(\sqrt{4+h}+2)}=\frac{1}{\sqrt{4+h}+2}$.

9. (7 points) Find a complete factorization of x^6-64.

   Solution: Note that x^6-64 is the difference of two squares. Hence $x^6-64=(x^3-8)(x^3+8)=(x-2)(x^2+2x+4)(x+2)(x^2-2x+4)$.

10. (7 points) Find a symbolic representation of $f\circ g(x)$ in the case where $f(x)=\sqrt{2x}-5$ and $g(x)=7-x$. Then find the implied domain of $f\circ g(x)$

    Solution: $f\circ g(x)=f(g(x))=f(7-x)=\sqrt{2(7-x)}-5=\sqrt{14-2x}-5$, and the implied domain is $x\leq 7$.

11. (7 points) The points $A=(0,0), B=(8,0)$, and $C=(3,6)$ are the vertices of triangle. Find the length of the longest side.

    Solution: The lengths of the three sides are $d_1=\sqrt{8^2}=8, \sqrt{6^2+3^2}=\sqrt{45}\approx 6.70$, and $\sqrt{5^2+6^2}=\sqrt{61}\approx 7.81$, so the length of the longest side is 8.

12. (7 points) What is the slope of the line joining the points $(-2, f(-2))$ and $(4, f(4))$, where f is the function defined by

$$f(x) = \begin{cases} x^2 - |x| & \text{if } x \le 2 \\ 3x - 2 & \text{if } x > 2 \end{cases}$$

Solution: The slope is $\frac{f(4)-f(-2)}{4-(-2)} = (10-2)/6 = 4/3$.

13. (7 points) Find the (implied) domain of the function $f(x) = \frac{\sqrt{x}}{x-3}$.

Solution: The domain is all real nonnegative real numbers except 3, ie, $[0, 3) \cup (3, \infty)$.

14. (12 points) Suppose the functions f and g are given by the table of values shown. Complete the table by calculating the values of $f \circ g(x)$ and $g \circ f(x)$ for each of the values of x in the table.

x	$f(x)$	$g(x)$	$f \circ g(x)$	$g \circ f(x)$
0	2	1		
1	3	5		
2	2	1		
3	5	4		
4	4	3		
5	2	0		

Solution:

x	$f(x)$	$g(x)$	$f \circ g(x)$	$g \circ f(x)$
0	2	1	3	1
1	3	5	2	4
2	2	1	3	1
3	5	4	4	0
4	4	3	5	3
5	2	0	2	1

15. (40 points) Evaluate each of the limits, or state that it does not exist.

(a) $\lim_{x \to \infty} \frac{x^2 + 9x - 11}{2x^2 - 4x + 23}$

Solution: The limit is just the ratio of the two coefficients of x^2, or $1/2$.

(b) $\lim_{z \to 2} \frac{z^3 - 8}{z - 2}$

Solution: The numerator factors into $(z-2)(z^2+2z+4)$, so the limit is just the value of (z^2+2z+4) at $z = 2$, which is 12.

(c) $\lim_{h \to 3} \frac{(2-h)^2 + (2+h)^2}{h^2 - 3h + 6}$

Solution: Just evaluate the numerator and denominator at $h = 3$ to get $\frac{1^2+5^2}{9-9+6} = 26/6 = 13/3$.

(d) $\lim_{x \to 3} \frac{x - 3}{x^2 - 9}$

Solution: The denominator factors into $(x-3)(x+3)$, so the limit is just the value of $\frac{1}{x+3}$ at $x = 3$, that is, $1/6$.

(e) $\lim_{x \to 2} f(x)$

where

$$f(x) = \begin{cases} (x-4)^2 & \text{if } x < 2 \\ 7 & \text{if } x = 2 \\ 5x - 6 & \text{if } x > 2 \end{cases}$$

Solution: Cover the left side of the graph to find the right limit, which is the value you get from the $5x-6$ piece, namely 4. Then cover the right half to get the left limit, $\lim_{x \to 2^-} (x-4)^2$, which is also 4. Hence the limit is 4.