October 4, 2017 **Name** ______________________________

The problems count as marked. The total number of points available is 169. The list below is an amalgamation of the four tests in the two sections, so the total number of points is greater than 169. Throughout this test, **show your work.**

1. (8 points) Find the exact value of the expression $|\pi - 7| + |2\pi - 10| + |3\pi - 8|$. Express your answer in the form $a\pi + b$ where a and b are integers.

   Solution: Solve each absolute value separately to get $7 - \pi$, $10 - 2\pi$ and $3\pi - 8$. Therefore, the sum is $7 - \pi + 10 - 2\pi + 3\pi - 8 = 17 - 8 = 9$.

2. (8 points) Find an equation for a line perpendicular to the line $5x - 2y = 7$ and which goes through the point $(-3, 9)$. Express your answer in slope-intercept form.

   Solution: The given line has slope $5/2$ so the one perpendicular has slope $-2/5$. Hence $y - 9 = (-2/5)(x + 3)$. Thus $y = -2x/5 + 39/5$.

3. (12 points) How many points (x, y) in the plane satisfy both $x^2 + y^2 = 25$ and $x^2 - 10x + y^2 - 24y = -105$? The correct answer is worth 2 points, the correct explanation is worth 10 points.

   Solution: 1. Both equations describe circles. The centers are 13 units apart and the sum of their radii is also 13.

4. (12 points) Suppose f, g and h are polynomials of degrees $7, 8$ and 9 respectively. What is the degree of the product $(f \circ g) \cdot (f + g + h)$, where $\circ$ means composition?

   Solution: 65. The degree of the composition is $7 \cdot 8 = 56$ and the degree of the sum is 9, so the product has degree $56 + 9 = 65$.

5. (30 points) Evaluate each of the limits indicated below.

   (a) $\lim\limits_{x\to 2} \dfrac{(2x)^3 - 64}{x^2 - 4}$

   Solution: Factor both numerator and denominator to get $\lim_{x\to 2} \frac{(2x-4)((2x)^2+8x+16)}{(x-2)(x+2)} = \lim_{x\to 2} \frac{2(2x)^2+8x+16)}{(x+2)} = \frac{2(16+16+16)}{4} = 24$.

   (b) $\lim\limits_{h\to 0} \dfrac{\sqrt{25+2h} - 5}{h}$

   Solution: Rationalize the numerator to get $\lim\limits_{h\to 0} \dfrac{\sqrt{25+2h} - 5}{h}$ $= \lim_{h\to 0} \frac{(\sqrt{25+2h}-5)(\sqrt{25+2h}+5)}{h(\sqrt{25+2h}+5)}$. This reduces to $\lim\limits_{h\to 0} \dfrac{(\sqrt{25+2h}-5)(\sqrt{25+2h}+5)}{h(\sqrt{25+2h}+5)} =$ $\dfrac{25+2h-25}{h(\sqrt{25+2h}+5)} = \lim\limits_{h\to 0} \dfrac{2}{\sqrt{25+2h}+5} = \dfrac{2}{2\sqrt{25}} = \dfrac{1}{5}$.

   (c) $\lim\limits_{h\to 0} \dfrac{(2+h)^3 - 8}{h}$.

   Solution: Expand the numerator to get

   $$\lim_{h\to 0} \frac{8 + 12h + 6h^2 + h^3 - 8}{h} = \lim_{h\to 0} \frac{12h + 6h^2 + h^3}{h} = \lim_{h\to 0} \frac{h(12 + 6h + h^2)}{h}$$

   $= \lim\limits_{h\to 0}(12+6h+h^2)$, and now the zero over zero problem has disappeared. So the limit is 12.

   (d) $\lim\limits_{x\to 1} \dfrac{x^2 - 4x + 3}{x^2 + x - 2}$

   Solution: Factor and eliminate the $x - 1$ from numerator and denominator to get

   $$\lim_{x\to 1} \frac{x-3}{x+2} = -2/3$$

   (e) $\lim\limits_{x\to 3} \dfrac{\frac{4}{x} - \frac{4}{3}}{x - 3}$

   Solution: The limit of both the numerator and the denominator is 0, so we must do the fractional arithmetic. The limit becomes

   $$\lim_{x\to 3} \frac{\frac{4(3-x)}{3x}}{(x-3)} = \lim_{x\to 3} \frac{-\frac{4(x-3)}{3x}}{x-3} = \lim_{x\to 3} \frac{-\frac{4}{3x}}{1} = -4/9.$$

   (f) $\lim\limits_{x\to\infty} \dfrac{3x^4 - 6}{(11 - 3x^2)^2}$

   Solution: The degrees of the numerator and denominator are both 4 so the limit is $3/9 = 1/3$.

(g) $\displaystyle\lim_{x\to-\infty} \frac{\sqrt{36x^2-3}}{9x-11}$

Solution: Divide both numerator and denominator by x to get $\lim_{x\to-\infty} \frac{\sqrt{36-3/x^2}}{9-11/x} = 6/9 = 2/3$ because the degree of the denominator is essentially the same as that of the numerator.

6. (21 points) Let

$$f(x) = \begin{cases} 2x+3 & \text{if } -1 < x \leq 0 \\ |x-3| & \text{if } 0 < x < 4 \\ 2 & \text{if } x = 4 \\ 5-x & \text{if } 4 < x \leq 6 \end{cases},$$

(a) What is the domain of f? Express your answer in interval notation.
Solution: $(-1, 0] \cup (0, 4) \cup \{4\} \cup (4, 6] = (-1, 6]$.

(b) What is $\lim\limits_{x \to 0^-} f(x)$?
Solution: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 2x + 3 = 3$.

(c) What is $\lim\limits_{x \to 0^+} f(x)$?
Solution: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} |x - 3| = 3$.

(d) Is f continuous at $x = 0$? Discuss why or why not.
Solution: f is continuous at $x = 0$ because $f(0) = \lim_{x \to 0} f(x) = 3$.

(e) What is $\lim\limits_{x \to 4^-} f(x)$?
Solution: $\lim_{x \to 4^-} f(x) = \lim_{x \to 4^-} |x - 3| = 1$.

(f) What is $\lim\limits_{x \to 4^+} f(x)$?
Solution: $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} |x - 3| = 1$.

(g) Is f continuous at $x = 4$? Discuss why or why not.
Solution: f is not continuous at $x = 4$ because $f(4) = 2 \neq \lim\limits_{x \to 2} f(x) = \lim\limits_{x \to 2} 5 - x = 1$.

7. (10 points) Find all the x-intercepts of the function

$$g(x) = (2x^2 - 1)^2(3x + 1) - (2x^2 - 1)(3x + 1).$$

Solution: Factor out the common terms to get $g(x) = (2x^2 - 1)(3x + 1)[(2x^2 - 1) - 1] = (2x^2 - 1)(3x + 1)(2x^2 - 2)$. Setting each factor equal to zero, we find the zeros are $x = -\sqrt{2}/2, x = \sqrt{2}/2$, $x = -1/3, x = 1$ and $x = -1$.

8. (20 points)

   (a) Find all solutions to $||x - 3| - 8| = 5$.

   Solution: First note that $|x - 3| - 8$ could be either 5 or -5. That is, $|x - 3| - 8 = 5$ or $|x - 3| - 8 = -5$. Thus either $|x - 3| = 13$ or $|x - 3| = 3$. Each of these has two solutions. The former, $x = -10$ and $x = 16$ and the later, $x = 6$ and $x = 0$.

   (b) Find the domain of the function $f(x) = \sqrt{||x - 3| - 8| - 5}$ and write your answer in interval form.

   Solution: Solve the inequality $||x - 3| - 8| - 5 \geq 0$. Change it to an equality and solve to get the four numbers $0, 6, -10$, and 16. Then use the test interval technique to get $(-\infty, -10] \cup [0, 6] \cup [16, \infty)$.

9. (30 points) Let $g(x) = \sqrt{\frac{(2x-15)(3x+17)}{x^2+x-6}}$. The sequence of steps below will enable you to find the (implied) domain of g. Let $r(x) = (g(x))^2 = \frac{(2x-15)(3x+17)}{x^2+x-6}$.

   (a) Find the zeros of r. That is, find all x for which $r(x) = 0$.

   Solution: Solve $2x - 15 = 0$ to get $x = 15/2$ and solve $3x + 17 = 0$ to get $x = -17/3$.

   (b) Find the value(s) of x for which r is undefined.

   Solution: Solve $x^2 + x - 6 = 0$ to get $x - 2 = 0$ or $x = 2$ and $x + 3 = 0$ to get $x = -3$.

   (c) Write as a union of intervals the set of real numbers that result by removing the values of x found in the first two parts.

   Solution: It is $(-\infty, -17/3) \cup (-17/3, -3) \cup (-3, 2) \cup (2, 15/2) \cup (15/2, \infty)$.

   (d) For each of the intervals in part (c), select a test point in the interval, and compute the sign (plus or minus) of r at that test point.

   Solution:

   (e) Express the domain of $g(x)$ as a union of intervals. Be sure to include or exclude the endpoints as appropriate.

   Solution: Since $r(x)$ is positive on the first third and fifth of the intervals, our answer is at least $(-\infty, -17/3) \cup (-3, 2) \cup (15/2, \infty)$. But the endpoints $15/2$ and $-17/3$ must be included also. Thus we have Domain of g:$(-\infty, -17/3] \cup (-3, 2) \cup [15/2, \infty)$.

10. (25 points) Let $f(x) = \sqrt{3x-2}$. Notice that $f(6) = \sqrt{18-2} = 4$.

    (a) Find the slope of the line joining the points $(6,4)$ and $(6+h, f(6+h))$, where $h \neq 0$. Note that $(6+h, f(6+h))$ is a point on the graph of f.

    Solution: $\frac{\sqrt{3(6+h)-2}-4}{6+h-6} = \frac{\sqrt{3(6+h)-2}-4}{h}$.

    (b) Compute $f(a+h), f(a)$, and finally $\frac{f(a+h)-f(a)}{h}$.

    Solution:

    (c) Finally compute the limit as h approaches 0 to find $f'(a)$.

    Solution: Since we get zero over zero, we recall that, in this case, we should rationalize the numerator.

$$
\begin{aligned}
\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0}\frac{\sqrt{3(a+h)-2}-\sqrt{3a-2}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{3(a+h)-2}-\sqrt{3a-2}}{h}\cdot\frac{\sqrt{3(a+h)-2}+\sqrt{3a-2}}{\sqrt{3(a+h)-2}+\sqrt{3a-2}} \\
&= \lim_{h\to 0}\frac{3(a+h)-2-(3a-2)}{h(\sqrt{3(a+h)-2}+\sqrt{3a-2})} \\
&= \lim_{h\to 0}\frac{3h}{h(\sqrt{3(a+h)-2}+\sqrt{3a-2})} \\
&= \lim_{h\to 0}\frac{3}{(\sqrt{3(a+h)-2}+\sqrt{3a-2})} \\
&= \frac{3}{2(\sqrt{3a-2})}
\end{aligned}
$$

    (d) Replace the a with 6 to find $f'(6)$.

    Solution: $f'(6) = 3\cdot 16^{-1/2}/2 = 3/8$

    (e) Use the information given and that found in (d) to find an equation for the line tangent to the graph of f at the point $(6,4)$.

    Solution: The line is $y-4 = 3(x-6)/8$, or $y = 3x/8 + 7/4$.

11. (20 points) Let $f(x) = x^2 - 2x$. Note that $f(3) = 3$

(a) Find the slope of the line joining the points $(3, 3)$ and $(3 + h, f(3 + h))$, where $h \neq 0$. Note that $(3 + h, f(3 + h))$ is a point on the graph of f.

Solution: The slope is $\frac{f(3+h)-f(3)}{3+h-3} = \frac{(3+h)^2-2(3+h)-3}{h}$.

(b) Evaluate and simplify $\frac{f(x+h)-f(x)}{h}$. Then find the limit of the expression as h approaches 0.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) - (x^2 - x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} \\ &= \lim_{h\to 0} \frac{h(2x + h - 2)}{h} = 2x - 2. \end{aligned}$$

(c) Replace the x with 3 in your answer to (b) to find $f'(3)$.

Solution: $f'(3) = 4$

(d) Use the information given and that found in (c) to find an equation for the line tangent to the graph of f at the point $(3, 3)$.

Solution: The line is $y - 3 = 4(x - 3)$, or $y = 4x - 9$.