October 2, 2019 **Name** ______________________________

The problems count as marked. The total number of points available is 171. Throughout this test, **show your work.** This is an amalgamation of the tests from sections 4 and 5.

1. (10 points) Find an equation for a line parallel to the line $2y + 3x = 12$ which passes through the point $(3, 5)$.

   Solution: The slope is $-3/2$ so the line in question is $y - 5 = -3(x - 3)/2$ which is $y = -3x/2 + 19/2$.

2. (10 points) Write the set of points that satisfy $||2x-15|-3| \leq 2$ using interval notation.

   Solution: First note that $||2x - 15| - 3| = 2$ gives rise to two equations, $|2x - 15| = 5$ and $|2x - 15| = 1$. Each of these splits into two linear equations, so we have $2x - 15 = -5, 2x - 15 = 5, 2x - 15 = -1, 2x - 15 = 1$, which in turn gives $2x = 10, 2x = 20, 2x = 14$ and $2x = 16$. So we have four branch points, $5, 10, 7$, and 8. Using the Test Interval Technique results in the solution $[5, 7] \cup [8, 10]$.

3. (25 points) The set C of points satisfying $x^2 - 24x + y^2 - 10y = -160$ is a circle.

   (a) Find the center and the radius of the circle C.

   Solution: Complete the square to get $(x - 12)^2 + (y - 5)^2 = 3^2$, a circle centered at $(12, 5)$ and radius 3.

   (b) There are two circles centered at $(0, 0)$ which are tangent to C. Find an equation for one of them.

   Solution: They are $x^2 + y^2 = 100$ and $x^2 + y^2 = 256$.

   (c) Find the point on the line $y = x$ that is closest to C.

   Solution: The slope of the line though the closest point (x, x) and $(12, 5)$ must be -1. Therefore $x = 17/2$.

4. (36 points) Evaluate each of the limits indicated below.

   (a) $\lim_{x\to 3} \dfrac{x^3-8}{x-2}$

   Solution: Factor both numerator and denominator to get $\lim_{x\to 2} \frac{(x-2)(x^2+2x+4)}{x-2} = \lim_{x\to 2} x^2+2x+4 = 12$.

   (b) $\lim_{x\to -2} \dfrac{x^3-3x^2-24x-28}{x^2-4}$

   (c) $\lim_{x\to -2} \dfrac{x^3-3x^2-24x-28}{x^3+2x^2-4x+8}$

   (d) $\lim_{x\to 1} \dfrac{x^3-5x^2+7x-3}{x^3-3x+2}$ Hint: think about why this is a zero over zero problem.

   Solution: Factor and eliminate the $(x-1)^2$ from numerator and denominator to get

   $$\lim_{x\to 1} \frac{x-3}{x+2} = -2/3$$

   (e) $\lim_{x\to 2} \dfrac{\frac{1}{3x}-\frac{1}{6}}{x-2}$

   Solution: The limit of both the numerator and the denominator is 0, so we must do the fractional arithmetic. The limit becomes

   $$\lim_{x\to 2} \frac{\frac{1}{3}[\frac{1}{x}-\frac{1}{2}]}{x-2} = \lim_{x\to 2} \frac{2-x}{x-2}\cdot\frac{1}{6x} = -\frac{1}{12}.$$

   (f) $\lim_{x\to 3} \dfrac{\sqrt{22-6x}-2}{x-3}$

   Solution: Rationalize the numerator to get

   $$\lim_{x\to 6} \frac{6x-36}{(x-6)(\sqrt{6x}+6)} = \frac{6}{12} = \frac{1}{2}$$

   (g) $\lim_{x\to -\infty} \dfrac{(2-x)(10+6x)^2}{(3-5x)(8+8x)^2}$

   Solution: The coefficient of the x^3 term in the numerator is -36 and the coefficient of the x^3 term in the denominator is -320, so the limit is $-36/-320 = 9/80$.

   (h) $\lim_{x\to \infty} \dfrac{\sqrt{4x^6-6}}{x^2+6}$

   Solution: The degree of the numerator (about 3) is greater than that of the denominator, so the limit is does not exist.

   (i) $\lim_{x\to \infty} \dfrac{\sqrt{4x^6-6}}{x^3+6}$

5. (12 points) Find the domain of the function

$$g(x) = \frac{\sqrt{(x+4)(2x-3)(3x-17)}}{x-6}.$$

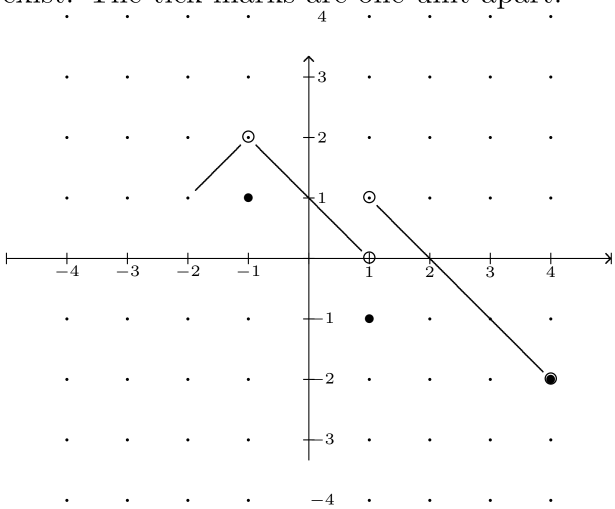
Express your answer as a union of intervals. That is, use interval notation.

Solution: Using the test interval technique, we see that the numerator is defined for when x belongs to $[-4, 3/2) \cup (17/3, \infty)$. The denominator is zero at $x = 6$, so it must be removed. Thus, the domain is $[-4, 3/2] \cup [17/3, 6) \cup (6, \infty)$.

6. (12 points) Let $H(x) = (x+1)(x^2 - 9) - (x-3)(3x+5)$. Find the zeros of the function.

Solution: Factor out the common terms to get $H(x) = (x+1)(x^2-9) - (x-3)(3x+5) = (x-3)[(x+1)(x+3) - (3x+5)]$. One factor is $x - 3$ and the other is $x^2 + x - 2 = (x+2)(x-1)$. So the zeros are $3, -2$, and 1.

7. (18 points) Consider the function F whose graph is given below. Evaluate each of the following expressions. Note: Enter 'DNE' if the limit does not exist. The tick marks are one unit apart.



(a) $\lim_{x\to -1^-} F(x) =$

Solution: 2

(b) $\lim_{x\to -1^+} F(x) =$

Solution: 2

(c) $\lim_{x\to -1} F(x) =$

Solution: 2

(d) $F(-1) =$

Solution: 1

(e) $\lim_{x\to 1^-} F(x) =$

Solution: 0

(f) $\lim_{x\to 1^+} F(x) =$

Solution: 1

(g) $\lim_{x\to 1} F(x) =$

Solution: dne

(h) $\lim_{x\to 3} F(x) =$

Solution: -1

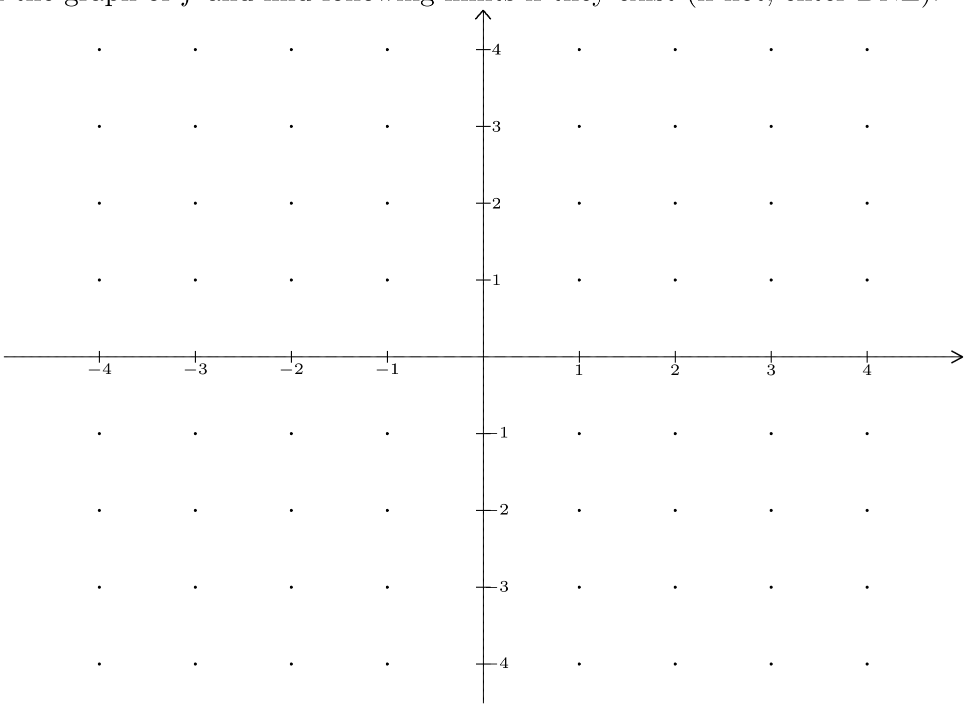
(i) $F(3) =$

Solution: -1

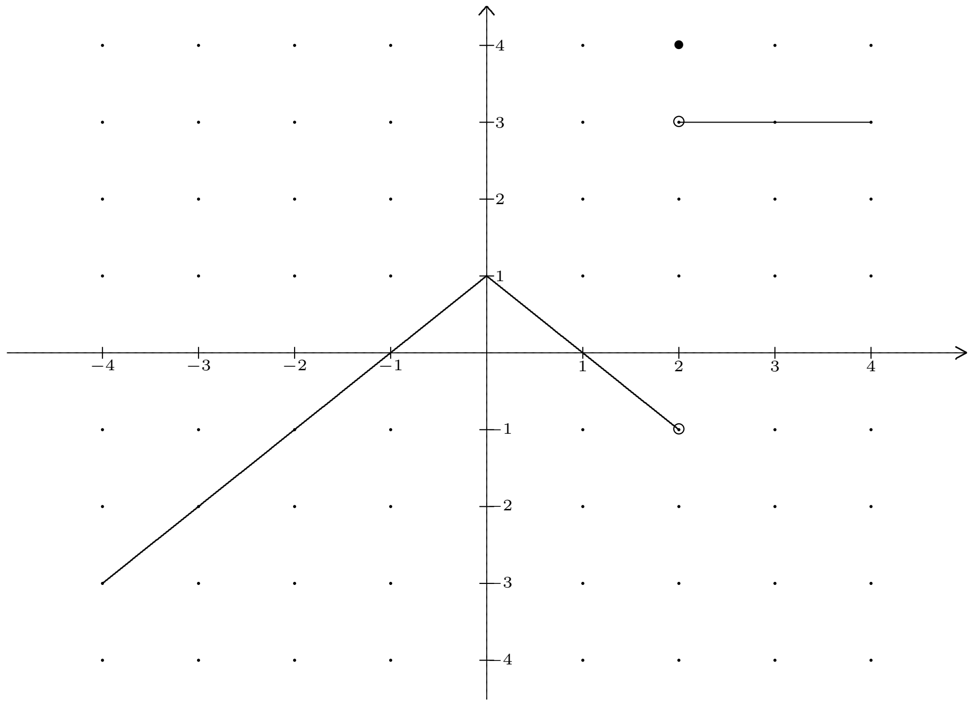
8. (18 points)

$$f(x) = \begin{cases} 3 & \text{if } 2 < x \leq 4 \\ 4 & \text{if } x = 2 \\ -x+1 & \text{if } 0 \leq x < 2 \\ x+1 & \text{if } -4 \leq x < 0 \end{cases}$$

Sketch the graph of f and find following limits if they exist (if not, enter DNE).



Solution: Sketch the graph of f and find following limits if they exist (if not, enter DNE).



(a) Express the domain of f in interval notation.
Solution: $[-4, 4]$.

(b) $\lim_{x\to 2^-} f(x)$
Solution: -1

(c) $\lim_{x\to 2^+} f(x)$
Solution: 3

(d) $\lim_{x\to 2} f(x)$
Solution: DNE

(e) $\lim_{x\to 0^-} f(x)$
Solution: 1

(f) $\lim_{x\to 0} f(x)$
Solution: 1

9. (12 points) Let $f(x) = (2x^2 - 3)^3(5x - 1) + 17x^5$, let $g(x) = (3x - 4)(2x^3)^2 - 2x^4$.

   (a) What is the degree of the polynomial $f + g$?

   Solution: 7

   (b) What is the degree of the polynomial $f \cdot g$?

   Solution: 14

   (c) Estimate within one tenth of a unit the value of $f(10000)/g(10000)$.

   Solution: Any answer between 3.3 and 3.4 works. See the next part.

   (d) Compute $\lim_{x\to\infty} \frac{f(x)}{g(x)}$.

   Solution: $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{(2x^2-3)^3(5x-1)+17x^5}{(3x-4)(2x^3)^2-2x^4} = \lim_{x\to\infty} \frac{40x^7}{12x^7} = 10/3$ because the degree of the denominator is the same as that of the numerator.

10. (15 points) Recall that the Intermediate Value Theorem guarantees that for any function f continuous over the interval $[a, b]$ and for any number M between $f(a)$ and $f(b)$, there exists a number c such that $f(c) = M$. The function $f(x) = \frac{1}{1+\frac{1}{x}}$ is continuous for all $x > 0$. Let $a = 1$.

    (a) Pick a number $b > 1$ (any choice is right), and then find a number M between $f(a)$ and $f(b)$.

    Solution: Suppose you picked $b = 2$. Then $f(a) = 1/2$ and $f(b) = 2/3$. You could choose $M = 3/5$.

    (b) Show that the conclusion to the Intermediate Value Theorem is satisfied by finding a number c in (a, b) such that $f(c) = M$.

    Solution: To solve $f(c) = 3/5$, write $\frac{1}{1+\frac{1}{x}} = 3/5$, from which we get $5 = 3 + 3/x$ and then $3/x = 2$, so $x = 3/2$. Indeed $3/2$ is between 1 and 2, as required.

11. (20 points) Let $f(x) = \frac{1}{x+1}$. Note that $f(0) = 1$.

(a) Find the slope of the line joining the points $(0, 1)$ and $(0 + h, f(0 + h)) = (h, f(h))$, where $h \neq 0$. Then find the limit as h approaches 0 to get $f'(0)$.

Solution: $\frac{f(h)-1}{h-0}$, which can be massaged to give $-\frac{1}{h+1}$. Thus $f'(0) = -1$.

(b) Evaluate and simplify $\frac{f(x+h)-f(x)}{h}$. Then find the limit of the expression as h approaches 0. In other words, find $f'(x)$.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+1} - \frac{1}{x+1}}{h} \\ &= \lim_{h\to 0} \frac{\frac{x+1-(x+h+1)}{(x+1)(x+h+1)}}{h} \\ &= \lim_{h\to 0} \frac{\frac{-h}{(x+1)(x+h+1)}}{h} \\ &= -\frac{1}{(x+1)^2}. \end{aligned}$$

(c) Replace the x with 0 in your answer to (b) to find $f'(0)$.

Solution: $f'(0) = -1$

(d) Use the information given and that found in (c) to find an equation in slope-intercept form for the line tangent to the graph of f at the point $(0, 1)$.

Solution: The line is $y - 1 = -1(x - 0)$, or $y = -x + 1$.

12. (12 points) Let

$$f(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x < 2 \\ 3 & \text{if } 2 \leq x \end{cases}$$

and let $g(x) = 2x - 1$.

(a) Build $g \circ f$.

Solution:

$$g \circ f(x) = 2f(x) - 1 = \begin{cases} -3 & \text{if } x \leq 0 \\ 1 & \text{if } 0 < x < 2 \\ 5 & \text{if } 2 \leq x \end{cases}$$

(b) Build $f \circ g$.

Solution:

$$f \circ g(x) = \begin{cases} -1 & \text{if } 2x - 1 \leq 0 \\ 1 & \text{if } 0 < 2x - 1 < 2 \\ 3 & \text{if } 2 \leq 2x - 1 \end{cases}$$

Therefore,

$$f \circ g(x) = \begin{cases} -1 & \text{if } x \leq 1/2 \\ 1 & \text{if } 1/2 < x < 3/2 \\ 3 & \text{if } 3/2 \leq x \end{cases}$$