September 24, 2004 **Name** ______________________________

The first 6 problems count 6 points each and the rest count as marked. The total number of points available is 137. Throughout this test, **show your work.**

1. What is the degree of the polynomial $p(x) = (x^3 - 1)^3(x^2 + 7)$?

   Solution: $(x^3 - 1)^3$ has degree 9 and $(x^2 + 7)$ has degree 2, so $p(x)$ has degree $9 + 2 = 11$.

2. Let P denote the midpoint of the line segment joining $(2, 3)$ and $(-6, 9)$. What is the distance from P to the point $(0, 3)$?

   Solution: $P = (\frac{2-6}{2}, \frac{3+9}{2}) = (-2, 6)$ so the distance is $d = \sqrt{(0 + 2)^2 + (3 - 6)^2} = \sqrt{13}$.

3. Compute the exact value of $|\pi + \sqrt{3} - 5| - |2\pi - \sqrt{3} + 1|$.

   Solution: $|\pi + \sqrt{3} - 5| = 5 - \pi - \sqrt{3}$ and $|2\pi - \sqrt{3} + 1| = 2\pi + 1 - \sqrt{3}$, so the difference is $5 - \pi - \sqrt{3} - (2\pi + 1 - \sqrt{3}) = 4 - 3\pi$.

4. Find the (implied) domain of

   $$f(x) = \frac{\sqrt{x-2}}{(x-5)(x-7)},$$

   and write your answer in interval notation.

   Solution: The domain D includes all real numbers greater than or equal to 2 except 7 and 5, which must be eliminated because they make the denominator zero. Thus, $D = [2, 5) \cup (5, 7) \cup (7, \infty)$.

5. Find all the x-intercepts of the function

   $$t(x) = (2x - 1)^3(x - 1)^2 - (2x - 1)^2(x - 1)^3.$$

   Solution: Factor the common stuff out to get $(2x-1)^2(x-1)^2[2x-1-(x-1)]$. Setting each of the three factors to zero yields $x = 1/2, x = 1$, and $x = 0$.

6. Find an equation for a line perpendicular to the line $3x - 4y = 7$ and which goes through the point $(-2, -3)$.

   Solution: The given line has slope $3/4$ so the one perpendicular has slope $-4/3$. Hence $y + 3 = (-4/3)(x + 2)$. Thus $y = -4x/3 - 17/3$.

7. (8 points) The line tangent to the graph of $y = e^{3x}$ at the point $(0, 1)$ has slope 3. What is the x-intercept of the line?

   Solution: The line is $y - 1 = 3(x - 0)$, so the x-intercept is $-1/3$.

8. (48 points) Compute each of the following limits.

(a) Let $f(x) = \begin{cases} x+2 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \\ 4 - x^2 & \text{if } x > 1 \end{cases}$

$\lim_{x \to 1} f(x)$

Solution: 3. Use the blotter test to see that $f(x)$ is close to 3 when x is close (but not equal) to 1. Alternatively, $\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x + 2 = 3$ and $\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} 4 - x^2 = 3$, so the limit is 3.

(b) $\lim_{x \to 0} \dfrac{x^2 - 2x}{x}$

Solution: Factor the numerator and cancel out the factor x to get $\lim_{x \to 0} \dfrac{x^2 - 2x}{x} = -2$.

(c) $\lim_{x \to 3} \dfrac{x^2 - 3x}{x^2 + x - 12}$

Solution: Factor and eliminate the common factor $x - 3$, then set $x = 3$ to get $3/(3 + 4) = 3/7$.

(d) $\lim_{x \to 2} |x^2 - \sqrt{16x - 7}|$

Solution: Just replace all the x's with the number 2 to get $|2^2 - \sqrt{16 \cdot 2 - 7}| = |4 - 5| = 1$.

(e) $\lim_{x \to 1} \dfrac{x^2 - 1}{x^3 - 1}$

Solution: Factor the denominator and cancel out the factor $x - 1$ to get $\lim_{x \to 1} \dfrac{x + 1}{x^2 + x + 1} = 2/3$.

(f) $\lim_{x \to 9} \dfrac{x - 9}{\sqrt{x} - 3}$

Solution: Rationalize the denominator to get $\frac{x-9}{\sqrt{x}-3} = \frac{(x-9)(\sqrt{x}+3)}{x-9}$ which has limit 6 as x approaches 9.

(g) $\lim_{x \to 1} \dfrac{\frac{1}{3x} - \frac{1}{3}}{x - 1}$

Solution: Do the fraction arithmetic to get $\frac{\frac{1}{3x} - \frac{1}{3}}{x-1} = \frac{\frac{1-x}{3x}}{\frac{x-1}{1}} = -\frac{1}{3x}$ which has limit $-1/3$ as x approaches 1.

(h) $\lim_{x \to \infty} \dfrac{2x^2}{(1 - x)^2}$

Solution: We are looking for the horizontal asymptote, which by the asymptote theorem is just $2/1 = 2$.

9. (15 points) Let $k(x) = x^2 - x$. Evaluate and simplify $\frac{k(x+h)-k(x)}{h}$. Then find the limit of the expression as h approaches 0.

Solution:

$$\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{k(x+h)-k(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2-(x+h)-(x^2-x)}{h} \\
&= \lim_{h\to 0} \frac{x^2+2\cdot xh+h^2-x-h-x^2+x}{h} \\
&= \lim_{h\to 0} \frac{+2\cdot xh+h^2-h}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h-1)}{h} = 2x-1.
\end{aligned}$$

10. (30 points) Consider the rational function $r(x) = \frac{(x+1)^2(2x+5)}{4x^3-16x}$.

    (a) Estimate the value $r(1000)$. Does $r(x)$ have a horizontal asymptote? Determine the degrees of the numerator n and the denominator m.

    Solution: Yes, $r(1000) \approx 1/2$ because the horizontal asymptote is $y = 1/2$. Note that $m = n = 3$ for this rational function. The horizontal line whose value is the ratio of the coefficients of x^3 in the numerator and denominator, $y = 2/4 = 1/2$.

    (b) Factor the denominator completely. Determine the vertical asymptotes.

    Solution: $4x^3 - 16x = x(4x^2 - 16) = x(x-2)(x+2)$. Thus the vertical asymptotes are $x = 0$, $x = 2$, and $x = -2$.

    (c) Use the Test Interval Technique to solve the inequality $r(x) \geq 0$. Be sure to show your work, including the matrix of values of the factors at the test points.

    Solution: There are five branch points, two from the numerator, $x = -5/2$ and $x = -1$, and three from the denominator, $x = 0$, $x = 2$, and $x = -2$. As we move past each of these among the six intervals determined by the branch points we find sign changes at all except the -1 (why?). So the answer is $(-\infty, -5/2) \cup (-2, -1) \cup (-1, 0) \cup (2, \infty)$ plus the numbers $x = -5/2$ and $x = -1$. So we can write the answer as $(-\infty, -5/2] \cup (-2, 0) \cup (2, \infty)$.