July 14, 2005 **Name** ______________________________

The first 6 problems count 6 points each and the rest count as marked. The total number of points available is 137. Throughout this test, **show your work.**

1. What is the degree of the polynomial $p(x) = (x^2 - 1)^3(x^5 - 7)$?

   Solution: $(x^2 - 1)^3$ has degree 6 and $(x^5 + 7)$ has degree 5, so $p(x)$ has degree $6 + 5 = 11$.

2. Let P denote the midpoint of the line segment joining $(4, 3)$ and $(-6, 9)$. What is the distance from P to the point $(0, 3)$?

   Solution: $P = (\frac{4-6}{2}, \frac{3+9}{2}) = (-1, 6)$ so the distance is $d = \sqrt{(0 + 1)^2 + (3 - 6)^2} = \sqrt{10}$.

3. Compute the exact value of $|4\pi - 5\sqrt{2}| + |4\pi - 13| - |5\sqrt{2} - 8|$.

   Solution: $|4\pi - 5\sqrt{2}| = 4\pi - 5\sqrt{2}$, because $4\pi - 5\sqrt{2} > 0$, $|4\pi - 13| = 13 - 4\pi$ because $4\pi - 13 < 0$ and $|5\sqrt{2} - 8| = 8 - 5\sqrt{2}$ because $5\sqrt{2} - 8 < 0$. Thus $|4\pi - 5\sqrt{2}| + |4\pi - 13| - |5\sqrt{2} - 8| = 4\pi - 5\sqrt{2} + 13 - 4\pi - (8 - 5\sqrt{2}) = 4\pi - 5\sqrt{2} + 13 - 4\pi - 8 + 5\sqrt{2} = 5$.

4. Find the (implied) domain of

   $$f(x) = \frac{\sqrt{x - 6}}{(x - 2)(x - 9)},$$

   and write your answer in interval notation.

   Solution: The domain D includes all real numbers greater than or equal to 6 except 2 and 9, which must be eliminated because they make the denominator zero. But the number 2 is less than 6 so we need not be concerned about it. Thus, $D = [6, 9) \cup (9, \infty)$.

5. Find all the x-intercepts of the function

   $$t(x) = (2x - 1)^3(x - 1)^2 - (2x - 1)^2(x - 1)^3.$$

   Solution: Factor the common stuff out to get $(2x-1)^2(x-1)^2[2x-1-(x-1)]$. Setting each of the three factors to zero yields $x = 1/2, x = 1$, and $x = 0$.

6. Find an equation for a line perpendicular to the line $3x - 4y = 7$ and which goes through the point $(-2, -5)$.

   Solution: The given line has slope $3/4$ so the one perpendicular has slope $-4/3$. Hence $y + 5 = (-4/3)(x + 2)$. Thus $y = -4x/3 - 23/3$.

7. (8 points) The line tangent to the graph of $y = e^{4x}$ at the point $(0, 1)$ has slope 4. What is the x-intercept of the line? Hint: recall the x-intercept is the point where the line crosses the x-axis.

   Solution: The line is $y - 1 = 4(x - 0)$, so the x-intercept is $-1/4$.

8. (48 points) Compute each of the following limits.

   (a) Let $f(x) = \begin{cases} x+2 & \text{if } x < 2 \\ 3 & \text{if } x = 2 \\ 8 - x^2 & \text{if } x > 2 \end{cases}$

   $\lim_{x\to 2} f(x)$

   Solution: The limit is 4. Use the blotter test to see that $f(x)$ is close to 4 when x is close (but not equal) to 2. Alternatively, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x + 2 = 4$ and $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 8 - x^2 = 4$, so the limit is 4.

   (b) $\lim_{x\to 0} \frac{x^2 - 3x}{x}$

   Solution: Factor the numerator and cancel out the factor x to get $\lim_{x\to 0} \frac{x^2 - 3x}{x} = -3$.

   (c) $\lim_{x\to 3} \frac{x^2 - 3x}{x^2 + x - 12}$

   Solution: Factor and eliminate the common factor $x - 3$, then set $x = 3$ to get $3/(3 + 4) = 3/7$.

   (d) $\lim_{x\to 1} \frac{x^2 - 1}{x^3 - 1}$

   Solution: Factor the denominator and cancel out the factor $x - 1$ to get $\lim_{x\to 1} \frac{x + 1}{x^2 + x + 1} = 2/3$.

   (e) $\lim_{x\to 9} \frac{x - 9}{\sqrt{x} - 3}$

   Solution: Rationalize the denominator to get $\frac{x-9}{\sqrt{x}-3} = \frac{(x-9)(\sqrt{x}+3)}{x-9}$ which has limit 6 as x approaches 9.

   (f) $\lim_{x\to 1} \frac{\frac{1}{3x} - \frac{1}{3}}{x - 1}$

   Solution: Do the fraction arithmetic to get $\frac{\frac{1}{3x} - \frac{1}{3}}{x-1} = \frac{\frac{1-x}{3x}}{\frac{x-1}{1}} = -\frac{1}{3x}$ which has limit $-1/3$ as x approaches 1.

(g) $\lim_{h\to 0} \frac{(3+h)^3 - 27}{h}$. Hint: you will have to work out the expanded form of $(3+h)^3$.

Solution: The expanded form of $(3+h)^3$ is $27 + 27h + 9h^2 + h^3$ so the limit is

$$\begin{aligned}
\lim_{h\to 0} \frac{27 + 27h + 9h^2 + h^3 - 27}{h} &= \lim_{h\to 0} \frac{+27h + 9h^2 + h^3}{h} \\
&= \lim_{h\to 0} \frac{h(27 + 9h + h^2)}{h} \\
&= \lim_{h\to 0} (27 + 9h + h^2) \\
&= 27.
\end{aligned}$$

(h) $\lim_{x\to\infty} \frac{3x^2}{(1-2x)^2}$

Solution: We are looking for the horizontal asymptote, which by the asymptote theorem is just $a_2/b_2 = 3/4$.

9. (15 points) Let $k(x) = x^2 - x$. Evaluate and simplify $\frac{k(x+h)-k(x)}{h}$. Then find the limit of the expression as h approaches 0.

Solution:

$$\begin{aligned}
k'(x) &= \lim_{h\to 0} \frac{k(x+h) - k(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - (x+h) - (x^2 - x)}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2\cdot xh + h^2 - x - h - x^2 + x}{h} \\
&= \lim_{h\to 0} \frac{+2\cdot xh + h^2 - h}{h} \\
&= \lim_{h\to 0} \frac{h(2x + h - 1)}{h} = 2x - 1.
\end{aligned}$$

10. (30 points) Consider the rational function $r(x) = \frac{(x+1)^2(2x+5)}{4x^3-16x}$.

(a) Estimate the value $r(1000)$. Does $r(x)$ have a horizontal asymptote? Determine the degrees of the numerator n and the denominator m.

Solution: Yes, $r(1000) \approx 1/2$ because the horizontal asymptote is $y = 1/2$. Note that $m = n = 3$ for this rational function. The horizontal line whose value is the ratio of the coefficients of x^3 in the numerator and denominator, $y = 2/4 = 1/2$.

(b) Factor the denominator completely. Determine the vertical asymptotes.

Solution: $4x^3 - 16x = x(4x^2 - 16) = x(x-2)(x+2)$. Thus the vertical asymptotes are $x = 0,\ x = 2$, and $x = -2$.

(c) Use the Test Interval Technique to solve the inequality $r(x) \geq 0$. Be sure to show your work, including the matrix of values of the factors at the test points.

Solution: There are five branch points, two from the numerator, $x = -5/2$ and $x = -1$, and three from the denominator, $x = 0,\ x = 2$, and $x = -2$. As we move past each of these among the six intervals determined by the branch points we find sign changes at all except the -1 (why?). So the answer is $(-\infty, -5/2) \cup (-2, -1) \cup (-1, 0) \cup (2, \infty)$ plus the numbers $x = -5/2$ and $x = -1$. So we can write the answer as $(-\infty, -5/2] \cup (-2, 0) \cup (2, \infty)$.