September 27, 2007 **Name** ______________________________

The problems count as marked. The total number of points available is 139. Throughout this test, **show your work.**

1. (40 points) Evaluate each of the limits indicated below.

   (a) $\lim_{x\to 4} \frac{\frac{2}{x} - \frac{1}{2}}{x-4}$

   Solution: Find a common denominator and eliminate the common terms to get $\lim_{x\to 4} -\frac{4-x}{4-x} \cdot \frac{1}{2x} = -1/8$.

   (b) $\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$

   Solution: Rationalize the numerator to get $\lim_{x\to 16} \frac{x-16}{(x-16)(\sqrt{x}+4)} = \lim_{x\to 16} 1/(\sqrt{x}+4) = 1/8$.

   (c) $\lim_{x\to 0} \frac{x^3+2x^2}{x^2}$

   Solution: Factor and eliminate the x^2 to get

   $$\lim_{x\to 0} x + 2 = 2$$

   (d) $\lim_{x\to -2} \frac{x^3+8}{x+2}$

   Solution: Factor $x^3 + 8 = (x+2)(x^2 - 2x + 4)$ and eliminate the $x + 2$ in both numerator and denominator to get $\lim_{x\to -2} x^2 - 2x + 4 = 12$.

   (e) $\lim_{x\to\infty} \frac{11+5x}{\sqrt{9x^2-3}}$

   Solution: $\lim_{x\to\infty} \frac{11+5x}{\sqrt{9x^2-3}} = \lim_{x\to\infty} \frac{11/x+5x/x}{\sqrt{9x^2/x^2-3/x^2}} = \frac{5}{3}$ because the degree of the denominator is essentially the same as that of the numerator.

For problems (f) through (k), let

$$f(x) = \begin{cases} 7 - x & \text{if } x < 0 \\ 10 & \text{if } x = 0 \\ (x+1)(x+7) & \text{if } 0 \leq x < 3 \\ 30 & \text{if } 3 \leq x \end{cases}$$

(f) $\lim_{x \to 0^-} f(x)$

Solution: 7

(g) $\lim_{x \to 0^+} f(x)$

Solution: 7

(h) $\lim_{x \to 0} f(x)$

Solution: 7

(i) $\lim_{x \to 3^-} f(x)$

Solution: 40

(j) $\lim_{x \to 3^+} f(x)$

Solution: 30

(k) $\lim_{x \to 3} f(x)$

Solution: DNE

2. (10 points) When $|2 - 4\pi - 3\sqrt{2}| + |4\sqrt{2} + 8 - 2\pi| + |6 - 6\pi - \sqrt{8}|$ is expressed in the form $a + b\sqrt{2} + c\pi$, where a, b, and c are integers, what are the values of a, b, and c? No points for a decimal approximation.

   Solution: $|2-4\pi-3\sqrt{2}|+|4\sqrt{2}+8-2\pi|+|6-6\pi-\sqrt{8}| = -(2-4\pi-3\sqrt{2}) + 4\sqrt{2}+8-2\pi-(6-6\pi-\sqrt{8}) = -2+8-6+(3\sqrt{2}+4\sqrt{2}+2\sqrt{2})+(4\pi-2\pi+6\pi) = 9\sqrt{2}+8\pi$, so $a = 0$, $b = 9$ and $c = 8$.

3. (21 points) Consider the function whose properties are displayed.

a	-1	0	1	2	3	4
$\lim_{x\to a^-} f(x)$	1	1	1	3	2	3
$\lim_{x\to a^+} f(x)$	1	2	1	3	2	3
$f(a)$	1	2	-1	1	4	3
$\lim_{x\to a^-} g(x)$	4	1	3	3	1	4
$\lim_{x\to a^+} g(x)$	1	2	0	3	1	4
$g(a)$	1	-1	3	DNE	DNE	4

Using the table above calculate the limits below. Enter 'DNE' if the limit doesn't exist OR if limit can't be determined from the information given.

(a) $\lim_{x\to 0^+}[f(x)+g(x)]$
Solution: 4

(b) $\lim_{x\to 0^-}[f(x)+g(x)]$
Solution: 2

(c) $\lim_{x\to 2}[f(x)+g(x)]$
Solution: 6

(d) $(f+g)(4)$
Solution: $3+4=7$.

(e) $f\circ g\circ f(-1)$
Solution: $f\circ g\circ f(-1) = f\circ g(1) = f(3) = 4$

(f) Find all points (in the table) at which f is continuous.
Solution: $x=-1$ and $x=4$.

(g) Find all points (in the table) at which g is continuous.
Solution: $x=4$.

4. (18 points) Find the (implied) domain of each of the functions given below. Write your answers in interval notation.

   (a) $f(x) = \sqrt{(x-2)(x-3)} - \sqrt{(x-5)(x-7)}$.

   Solution: We can think of this as two problems. Since f is the difference of the two functions, $h(x) = \sqrt{(x-2)(x-3)}$ and $k(x) = \sqrt{(x-5)(x-7)}$, we can find the domains of each of these and take the real numbers common to both domains. The domain D_h of the first one is all the real numbers except $(2, 3)$ and the domain D_k of the second one is all the real numbers except $(5, 7)$. So the domain D of f is $D = (-\infty, 2] \cup [3, 5] \cup [7, \infty)$.

   (b) $g(x) = (2x^2 + 5x - 12)^{-1}$.

   Solution: Note that $2x^2 + 5x - 12$ factors into $(2x - 3)(x + 4)$. Therefore the zeros of the denominator are $x = 3/2$ and $x = -4$. The domain of g is therefore all real numbers except those two, namely $(-\infty, -4) \cup (-4, 3/2) \cup (3/2, \infty)$.

5. (25 points) Let $f(x) = \sqrt{2x+1}$. Notice that $f(4) = \sqrt{9} = 3$.

   (a) Find the slope of the line joining the points $(4, 3)$ and $(4+h, f(4+h))$, where $h \neq 0$. Note that $(4+h, f(4+h))$ is a point on the graph of f.

   Solution: $\frac{\sqrt{2(4+h)+1}-3}{4+h-4} = \frac{\sqrt{2(4+h)+1}-3}{h}$.

   (b) Compute $f(a+h)$, $f(a)$, and finally $\frac{f(a+h)-f(a)}{h}$.

   Solution: see below

   (c) Finally compute the limit as h approaches 0 to find $f'(a)$.

   Solution: Since we get zero over zero, we recall that, in this case, we should rationalize the numerator.

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(a+h)-f(a)}{h} &= \lim_{h\to 0} \frac{\sqrt{2(a+h)+1}-\sqrt{2a+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(a+h)+1}-\sqrt{2a+1}}{h} \cdot \frac{\sqrt{2(a+h)+1}+\sqrt{2a+1}}{\sqrt{2(a+h)+1}+\sqrt{2a+1}} \\
&= \lim_{h\to 0} \frac{2(a+h)+1-(2a+1)}{h(\sqrt{2(a+h)+1}+\sqrt{2a+1})} \\
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2(a+h)+1}+\sqrt{2a+1})} \\
&= \lim_{h\to 0} \frac{2}{(\sqrt{2(a+h)+1}+\sqrt{2a+1})} \\
&= \frac{2}{2(\sqrt{2a+1})} = \frac{1}{\sqrt{2a+1}}
\end{aligned}
$$

   (d) Replace the a with 4 to find $f'(4)$.

   Solution: $f'(4) = 2 \cdot 9^{-1/2}/2 = 1/3$

6. (32 points) Given three functions, $h(x) = 2x$,

$$g(x) = \begin{cases} x^2 + 1 & \text{if } x > 3 \\ 4 - x & \text{if } x \leq 3 \end{cases} \quad \text{and} \quad f(x) = \begin{cases} \sqrt{x+3} & \text{if } x \geq 2 \\ 2x - 1 & \text{if } x < 2 \end{cases}$$

Note that $f \circ g \circ h(-2) = f \circ g(h(-2)) = f \circ g(-4) = f(8) = \sqrt{11}$.

(a) Complete the following table.

x	$h(x)$	$g \circ h(x)$	$f \circ g \circ h(x)$
-2	-4	8	$\sqrt{11}$
3/2			
	10		
		10	
			3

Solution:

x	$h(x)$	$g \circ h(x)$	$f \circ g \circ h(x)$
-2	-4	8	$\sqrt{11}$
3/2	3	1	1
5	10	101	$\sqrt{104}$
-3	-6	10	$\sqrt{13}$
-1	-2	6	3

(b) Find all solutions to $f \circ g \circ h(x) = 3$.

Solution: $g(2x)$ could be 6 since $f(6) = \sqrt{9} = 3$. This equation leads to $2x = -2$ and $x = -1$.

(c) Find a symbolic representation of $g \circ h(x)$.

Solution:

$$g \circ h(x) = \begin{cases} (2x)^2 + 1 & \text{if } 2x > 3 \\ 4 - 2x & \text{if } 2x \leq 3 \end{cases}$$

and this simplifies to

$$g \circ h(x) = \begin{cases} 4x^2 + 1 & \text{if } x > 3/2 \\ 4 - 2x & \text{if } x \leq 3/2 \end{cases}$$