February 12, 2009 **Name** ______________________________

The problems count as marked. The total number of points available is 180. Throughout this test, **show your work.**

1. (8 points) Find the exact value of the expression $|10-3\sqrt{5}|-|2\sqrt{5}-4|-|\sqrt{5}-6|$. Express your answer in a very simple form.

   Solution: Solve each absolute value separately to get $10-3\sqrt{5}, 2\sqrt{5}-4$ and $-(\sqrt{5}-6)$. Therefore, the value is $10-3\sqrt{5}-(2\sqrt{5}-4)-(-(\sqrt{5}-6)) = 10-3\sqrt{5}-2\sqrt{5}+4+\sqrt{5}-6 = 10+4-6-(3+2-1)\sqrt{5} = 8-4\sqrt{5}$.

2. (8 points) Find an equation for a line perpendicular to the line $3x-2y=7$ and which goes through the point $(-3,5)$.

   Solution: The given line has slope $3/2$ so the one perpendicular has slope $-2/3$. Hence $y-5=(-2/3)(x+3)$. Thus $y=-2x/3+3$.

3. (52 points) Evaluate each of the limits indicated below.

   (a) $\lim_{x\to 1} \frac{x^2+2x-3}{x^2-6x+5}$
   Solution: Factor and eliminate the $x-1$ from numerator and denominator to get

   $$\lim_{x\to 1} \frac{x+3}{x-5} = 4/-4 = -1$$

   (b) $\lim_{x\to 3} \frac{\frac{2}{x}-\frac{2}{3}}{x-3}$
   Solution: The limit of both the numerator and the denominator is 0, so we must do the fractional arithmetic. The limit becomes

   $$\lim_{x\to 3} \frac{\frac{2(3-x)}{2x}}{2(x-3)} = \lim_{x\to 3} \frac{-\frac{2(x-3)}{3x}}{x-3} = \lim_{x\to 3} \frac{-\frac{2}{3x}}{1} = -2/9.$$

   (c) $\lim_{x\to -\infty} \frac{|\ 18x-3\ |}{6x-11}$
   Solution: Divide both numerator and denominator by x to get

   $$\lim_{x\to -\infty} \frac{3/x-18x/x}{11/x-6x/x} = 18/-6 = -3$$

   because the degree of the denominator is essentially the same as that of the numerator.

(d) $\lim_{x\to\infty} \dfrac{6x^4 - 3}{(11 - 3x^2)^2}$

Solution: The degrees of the numerator and denominator are both 4 so the limit is $6/9 = 2/3$.

(e) $\lim_{x\to -1} \dfrac{x^3 + 1}{x^2 - 1}$

Solution: Factor both numerator and denominator to get $\lim_{x\to -1} \frac{x^3+1}{x^2-1} = \lim_{x\to -1} \frac{(x+1)(x^2-x+1)}{(x-1)(x+1)} = \lim_{x\to -1} \frac{x^2-x+1}{x-1} = -3/2$

(f) $\lim_{h\to 0} \dfrac{(1+h)^3 - 1}{h}$.

Solution: Expand the numerator to get $\lim_{h\to 0} \frac{1+3h+3h^2+h^3-1}{h} = \lim_{h\to 0} \frac{3h+3h^2+h^3}{h} = \lim_{h\to 0} \frac{h(3+3h+h^2)}{h} = \lim_{h\to 0}(3+3h+h^2)$, and now the zero over zero problem has disappeared. So the limit is 3.

For problems (g) through (m), let

$$f(x) = \begin{cases} -2 & \text{if } x < 0 \\ 2x^2 - 2 & \text{if } 0 \leq x < 2 \\ 3 & \text{if } x = 2 \\ 10 - 2x & \text{if } x > 2 \end{cases}$$

(g) $\lim_{x \to 2^-} f(x)$
Solution: 6

(h) $\lim_{x \to 2^+} f(x)$
Solution: 6

(i) $\lim_{x \to 2} f(x)$
Solution: 6

(j) $\lim_{x \to 0^-} f(x)$
Solution: -2

(k) $\lim_{x \to 0^+} f(x)$
Solution: -2

(l) $\lim_{x \to 0} f(x)$
Solution: -2

(m) $f(0)$
Solution: -2

4. (12 points) The demand curve for a certain item is given by $p = -x^2 - 8x + 100$ where x represents the quantity demanded in units of a thousand and p represents the price in dollars. The supply curve is given by $p = 4x + 20$. Find the equilibrium quantity and equilibrium price.

   Solution: Solve the two equations simultaneously by setting $-x^2 - 8x + 100$ equal to $4x + 20$. This results in the quadratic $x^2 + 12x - 80 = 0$, which we can solve by the quadratic formula. $x = \frac{-12 \pm \sqrt{144+4 \cdot 1 \cdot 80}}{2} \approx \frac{-12+21.54}{2} \approx 4.77$. The other root is negative. The corresponding p value is $4 \cdot 4.77 + 20 = 39.08$.

5. (10 points) Find all the x-intercepts of the function

$$g(x) = (2x^2 - 1)^2(3x + 1) - (2x^2 - 1)(3x + 1)^2.$$

   Solution: Factor out the common terms to get $g(x) = (2x^2 - 1)(3x + 1)[(2x^2 - 1) - (3x + 1) = (2x^2 - 1)(3x + 1)(2x^2 - 3x - 2)$. Setting each factor equal to zero, we find the zeros are $x = -\sqrt{2}/2, x = \sqrt{2}/2$, $x = -1/3, x = 2$ and $x = -1/2$.

6. (30 points) Let $g(x) = \sqrt{\frac{2x-7)(3x+4)}{x^2-6x+5}}$. The sequence of steps below will enable you to find the (implied) domain of g. Let $r(x) = (g(x))^2 = \frac{(2x-7)(3x+4)}{x^2-6x+5}$.

   (a) Find the zeros of r.

   Solution: Solve $2x - 7 = 0$ to get $x = 7/2$ and solve $3x + 4 = 0$ to get $x = -4/3$.

   (b) Find the value(s) of x for which r is undefined.

   Solution: Solve $x^2 - 6x + 5 = 0$ to get $x - 5 = 0$ or $x = 5$ and $x - 1 = 0$ to get $x = 1$.

   (c) Write as a union of intervals the set of real numbers that result by removing the values of x found in the first two parts.

   Solution: It is $(-\infty, -4/3) \cup (-4/3, 1) \cup (1, 7/2) \cup (7/2, 5) \cup (5, \infty)$.

   (d) For each of the intervals in part 3, select a point in the interval, and compute the sign (plus or minus) of r at that test point.

   Solution:

   (e) Express the domain of $g(x)$ as a union of intervals. Be sure to include or exclude the endpoints as appropriate.

   Solution: Since $r(x)$ is positive on the first third and fifth of the intervals, our answer is at least $(-\infty, -4/3) \cup (1, 7/2) \cup (5, \infty)$. But the endpoints $7/2$ and $-4/3$ must be included also. Thus we have Domain of g: the set $(-\infty, -4/3] \cup (1, 7/2] \cup (5, \infty)$.

7. (20 points) Let $f(x) = x^2 - 2x$. Note that $f(2) = 0$

(a) Find the slope of the line joining the points $(2, 0)$ and $(2 + h, f(2 + h))$, where $h \neq 0$. Note that $(2 + h, f(2 + h))$ is a point on the graph of f. Then find the limit of the expression as h approaches 0 to compute $f'(2)$.

Solution: The slope is $\frac{f(2+h)-f(2)}{2+h-2} = \frac{(2+h)^2-2(2+h)-(2^2-2\cdot 2)}{h} = \frac{2h+h^2}{h}$. The limit of the expression as h goes to zero is 2.

(b) Evaluate and simplify $\frac{f(x+h)-f(x)}{h}$. Then find the limit of the expression as h approaches 0.

Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - k(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) - (x^2 - 2x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} \\ &= \lim_{h\to 0} \frac{h(2x + h - 2)}{h} = 2x - 2. \end{aligned}$$

(c) Replace the x with 2 to find $f'(2)$.

Solution: $f'(2) = 2$

(d) Use the information given and that found in (c) to find an equation for the line tangent to the graph of f at the point $(2, 0)$.

Solution: The line is $y - 0 = 2(x - 2)$, or $y = 2x - 4$.

8. (40 points) Below is a table of some of the values of two functions f and g and information about their some of their left-hand and right-hand limits. All the questions below refer to values of a in the set $\{-2,-1,0,1,2,3\}$.

a	$f(a)$	$g(a)$	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a^-} g(x)$	$\lim_{x\to a^+} g(x)$
-2	1	2	1	1	2	DNE
-1	0	1	2	2	1	1
0	2	-1	DNE	0	-1	-1
1	-1	0	-1	-1	2	-2
2	-2	-1	-2	-1	-1	-1
3	1	1	1	1	1	-1

(a) For which values of a does $\lim_{x\to a} f(x)$ exist?

Solution: All these problems involve looking up the answers in the table. Notice that the left and right limits of f are the same when $a = -2, -1, 1$, and 3.

(b) For which values of a does $\lim_{x\to a} g(x)$ exist?

Solution: The values are $-1, 0$, and 2

(c) For which values of a is $f(x)$ continuous?

Solution: The values are $-2, 1$, and 3.

(d) For which values of a is $g(x)$ continuous?

Solution: g is continuous when its limit and its value are the same, and this occurs when $a = -1, 0$ and 2.

(e) Find each of the following, if they exist.

i. $\lim_{x\to -1}[f(x)\cdot g(x)]$.
Solution: 2

ii. $f \circ g \circ f(1)$
Solution: -1

iii. $g \circ g \circ g(1)$
Solution: $g \circ g \circ g(1) = g \circ g(0) = g(-1) = 1$

iv. $g(\lim_{x\to -1} f(x))$.
Solution: $g(2) = -1$

(f) Find a value of a satisfying each of the equations. If more than one value exists, find them all.

i. $f \circ g(a) = 0$.
Solution: There are two values, 0 and 2.

ii. $g \circ f(a) = 0$.
Solution: There are two values, -2 and 3.

iii. $(f(a))^2 + (g(a))^2 = 5$.
Solution: There are three values,$-2, 0$, and 2.

iv. $(\lim_{x \to a} f(x))^2 + (\lim_{x \to a} g(x))^2 = 5$.
Solution: There are two values, -1 and 1.