February 16, 2011 **Name** ______________________________

The problems count as marked. The total number of points available is 155. Throughout this test, **show your work.**

1. (10 points) The points $(2, 6)$ and $(5, 3)$ belong a line L. Find an equation for the line perpendicular to L and passing through the point $(1, 1)$.

   Solution: The line L has slope -1 so the one perpendicular has slope 1. Hence the line in question is $y - 1 = 1(x - 1)$ or $y = x$.

2. (10 points) Find the exact value of $|2\pi - \sqrt{5} - 3\sqrt{2}| + 2\pi$. A decimal approximation is worth 1 point. Your answer may use radicals or symbol π.

   Solution: Since $2\pi - \sqrt{5} - 3\sqrt{2}$ is a negative number, $|2\pi - \sqrt{5} - 3\sqrt{2}| = -(2\pi - \sqrt{5} - 3\sqrt{2}) = \sqrt{5} + 3\sqrt{2} - 2\pi$. Adding 2π gives us $\sqrt{5} + 3\sqrt{2}$.

3. (30 points) Evaluate each of the limits indicated below.

   (a) $\displaystyle\lim_{x\to\infty} \frac{3x^4 - 6}{(11 - 3x^2)^2}$

   Solution: The degrees of the numerator is 4 while the degree of the denominator is 4, so the limit is $3/9 = 1/3$.

   (b) $\displaystyle\lim_{x\to 1} \frac{x^4 - 1}{x^2 - 1}$

   Solution: Factor both numerator and denominator to get $\lim_{x\to 1} \frac{x^4-1}{x^2-1} = \lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{(x^2-1)} = \lim_{x\to 1} \frac{x^2+1}{1} = 2$

   (c) $\displaystyle\lim_{x\to -3} \frac{x^2 + 2x - 3}{x^2 + 4x + 3}$

   Solution: Factor and eliminate the $x + 3$ from numerator and denominator to get

   $$\lim_{x\to -3} \frac{x - 1}{x + 1} = -4/-2 = 2$$

   (d) $\displaystyle\lim_{x\to 2} \frac{\frac{1}{4x} - \frac{1}{8}}{\frac{1}{3x} - \frac{1}{6}}$

   Solution: The limit of both the numerator and the denominator is 0, so we must either factor or do the fractional arithmetic. Factoring seems to work best. The limit becomes

   $$\lim_{x\to 2} \frac{\frac{1}{4}[\frac{1}{x} - \frac{1}{2}]}{\frac{1}{3}[\frac{1}{x} - \frac{1}{2}]} = \lim_{x\to 2} \frac{1}{4} \cdot \frac{3}{1} = \frac{3}{4}.$$

(e) $\lim_{x\to-\infty} \dfrac{|x^3|}{x^3 - x^2 + x - 1}$

Solution: Divide both numerator and denominator by x^3 to get $\lim_{x\to-\infty} \frac{|x^3|/x^3}{1-1/x+1/x^2-1/x^3}$, which approaches -1 as $x \to -\infty$.

(f) $\lim_{x\to 2} \dfrac{x-2}{\sqrt{8x}-4}$

Solution: Rationalize the denominator to get

$$\lim_{x\to 2} \frac{x-2}{8x-16}(\sqrt{8x}+4) = \frac{1}{8} \cdot 8 = 1$$

4. (12 points) Find the domain of the function

$$g(x) = \frac{\sqrt{x(8-x)}}{(x+1)(x-3)}.$$

Express your answer as a union of intervals. That is, use interval notation.

Solution: The numerator is defined for $0 \le x \le 8$, that is $[0, 8]$. The denominator is zero at $x = -1$ and at $x = 3$, so 3 must be removed from $[0, 8]$. Thus, the domain is $[0, 3) \cup (3, 8]$.

5. (12 points) Let $H(x) = (x^2-1)^2(5x+7)+(x^2-1)(5x+7)^2$. H is a polynomial of degree 5, and it has 5 zeros. Find all the zeros of H.

   Solution: Factor out the common terms to get $H(x) = (x^2-1)(5x+7)[x^2-1+5x+7]$. The zeros of x^2-1 are $x = \pm 1$ and the zero of $5x+7$ is $x = -7/5$. The quadratic x^2+5x+6 factors into $(x+3)(x+2)$, so its two zeros are $x = -3$ and $x = -2$.

6. (10 points) Suppose $p(x)$ is a polynomial of degree 4 and $q(x)$ is a polynomial of degree 3. What is the degree of the polynomial $H(x) = (x^2p(x)-1)^2 - (q(x)+x^2)^2 + x^8$? Write a sentence about your reasoning.

   Solution: We can reason as follows: $H(x)$ is the sum of three polynomials $(x^2p(x)-1)^2, -(q(x)+x^2)^2$ and x^8, and their degrees are, respectively $12, 6$ and 8, so H has degree 12.

7. (16 points) Let

$$f(x) = \begin{cases} |x-3| & \text{if } x < 2 \\ 5 & \text{if } x = 2 \\ (4-x)^2 & \text{if } x > 2 \end{cases},$$

   (a) What is $\lim_{x\to 2^-} f(x)$?
   Solution: $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} |x-3| = 1$.

   (b) What is $\lim_{x\to 2^+} f(x)$?
   Solution: For the limit to exist, the limit from the right must be 1, so $t = 1$ and $t = 3$ both work.

   (c) Is f continuous at $x = 2$?
   Solution: No. The limit does not exist.

   (d) What is $\lim_{x\to 1^-} f(x)$?
   Solution: f is continuous at all x except 2, so $\lim_{x\to 1^-} f(x) = |1-3| = 2$.

8. (20 points) Let $f(x) = x^2 - 2x$. Note that $f(3) = 3$

   (a) Find the slope of the line joining the points $(3, 3)$ and $(3 + h, f(3 + h))$, where $h \neq 0$. Note that $(3 + h, f(3 + h))$ is a point on the graph of f.

   Solution: The slope is $\frac{f(3+h)-f(3)}{3+h-3} = \frac{(3+h)^2-2(3+h)-3}{h}$.

   (b) Evaluate and simplify $\frac{f(x+h)-f(x)}{h}$. Then find the limit of the expression as h approaches 0.

   Solution:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^2 - 2(x+h) - (x^2 - x)}{h} \\ &= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - 2x - 2h - x^2 + 2x}{h} \\ &= \lim_{h\to 0} \frac{2xh + h^2 - 2h}{h} \\ &= \lim_{h\to 0} \frac{h(2x + h - 2)}{h} = 2x - 2. \end{aligned}$$

   (c) Replace the x with 3 in your answer to (b) to find $f'(3)$.

   Solution: $f'(3) = 4$

   (d) Use the information given and that found in (c) to find an equation for the line tangent to the graph of f at the point $(3, 3)$.

   Solution: The line is $y - 3 = 4(x - 3)$, or $y = 4x - 9$.

9. (20 points) For each condition listed, express in interval notation the set of all numbers that satisfy the condition. For example $1 \leq 2x - 3 < 7$ has solution the interval $[2, 5)$.

   (a) $x^2 \neq 9$
   Solution: You get $(-\infty, -3) \cup (-3, 3) \cup (3, \infty)$ when you 'pluck' out the numbers 3 and -3 from the number line.

   (b) $x^2 \geq 4$
   Solution: $(-\infty, -2] \cup [2, \infty)$

   (c) $(x - 2)(x + 3) \leq 0$
   Solution: Use the test interval technique to get $[-3, 2]$

   (d) $| 2x + 3 | \geq 9$
   Solution: The inequality is equivalent to $2x + 3 \geq 9$ or $2x + 3 \leq -9$. This is the same as $2x \geq 6$ or $2x \leq -12$, so we have $(-\infty, -6] \cup [3, \infty)$

10. (15 points) Recall that the Intermediate Value Theorem guarantees that for any function f continuous over the interval $[a, b]$ and for any number M between $f(a)$ and $f(b)$, there exists a number c such that $f(c) = M$. The function $f(x) = \frac{1}{1+\frac{1}{x}}$ is continuous for all $x > 0$. Let $a = 1$.

    (a) Pick a number $b > 1$ (any choice is right), and then find a number M between $f(a)$ and $f(b)$.
    Solution: Suppose you picked $b = 2$. Then $f(a) = 1/2$ and $f(b) = 2/3$. You could choose $M = 3/5$ between them.

    (b) Show that the conclusion to the Intermediate Value Theorem is satisfied by finding a number c in (a, b) such that $f(c) = M$.
    Solution: To solve $f(c) = 3/5$, write $\frac{1}{1+\frac{1}{x}} = 3/5$, from which we get $5 = 3 + 3/x$ and then $3/x = 2$, so $x = 3/2$. Indeed $3/2$ is between 1 and 2, as required.