October 2, 2012 **Name** ________________________________

The problems count as marked. The total number of points available is 169. Throughout this test, **show your work.**

1. (10 points) Find the exact value of $|5\sqrt{2}-7|+|1-4\sqrt{2}|-|9\sqrt{2}-11|$.

   Solution: $|5\sqrt{2}-7|+|1-4\sqrt{2}|-|9\sqrt{2}-11| = (5\sqrt{2}-7)+(4\sqrt{2}-1)-(9\sqrt{2}-11) = 3$.

2. (10 points) The points $(2,k)$ and $(5,5)$ belong to the line perpendicular to the line $6x-2y=7$. Find the value of k.

   Solution: The given line has slope 3 so the one perpendicular has slope $-1/3$. Hence $\frac{k-5}{2-5} = -1/3$. Solving, we get $k=6$.

3. (35 points) Evaluate each of the limits indicated below.

   (a) $\lim_{x\to\infty} \dfrac{3x^6+x^4-6}{(11-3x^3)^2}$

   Solution: The degree of both the numerator and the denominator is 6, so the limit is limit is $3/9 = 1/3$.

   (b) $\lim_{x\to 2} \dfrac{x^2-4}{x^4-16}$

   Solution: Factor the denominator to get $\lim_{x\to 2} \frac{x^2-4}{(x^2-4)(x^2+4)} = \lim_{x\to 2} \frac{1}{(x^2+4)} = \lim_{x\to 2} \frac{1}{8} = 1/8$

   (c) $\lim_{h\to 0} \dfrac{(1+h)^2-1}{h}$.

   Solution: Expand the numerator to get

   $$\lim_{h\to 0} \frac{1+2h+h^2-1}{h} = \lim_{h\to 0} \frac{2h+h^2}{h} = \lim_{h\to 0} \frac{h(2+h)}{h}$$

   $= \lim\limits_{h\to 0}(2+h)$, and now the zero over zero problem has disappeared. So the limit is 2.

   (d) $\lim_{x\to 1} \dfrac{x^2-4x+3}{x^2+x-2}$

   Solution: Factor and eliminate the $x-1$ from numerator and denominator to get

   $$\lim_{x\to 1} \frac{x-3}{x+2} = -2/3$$

(e) $\lim_{x\to 2} \frac{\frac{1}{3x} - \frac{1}{6}}{\frac{1}{2x} - \frac{1}{4}}$

Solution: The limit of both the numerator and the denominator is 0, so we must either factor or do the fractional arithmetic. Factoring seems to work best. The limit becomes

$$\lim_{x\to 2} \frac{\frac{1}{3}[\frac{1}{x} - \frac{1}{2}]}{\frac{1}{2}[\frac{1}{x} - \frac{1}{2}]} = \lim_{x\to 2} \frac{1}{3} \cdot \frac{2}{1} = \frac{2}{3}.$$

(f) $\lim_{x\to -\infty} \frac{\sqrt{36x^2 - 3x}}{9x - 11}$

Solution: Divide both numerator and denominator by x to get $\lim_{x\to -\infty} \frac{-\sqrt{36-3/x}}{9-11/x} = 6/9 = -2/3$ because the degree of the denominator is essentially the same as that of the numerator.

(g) $\lim_{x\to 2} \frac{\sqrt{8x} - 4}{x - 2}$

Solution: Rationalize the numerator to get

$$\lim_{x\to 2} \frac{8x - 16}{x - 2} \frac{1}{\sqrt{8x} + 4} = 8 \cdot \frac{1}{8} = 1$$

4. (30 points) A topless box is constructed from a rectangular piece of cardboard that measures 16 inches by 12 inches. An x by x square is cut from each of the four corners, and the sides are then folded upwards to build the box.

   (a) Express the volume V as a function of x.
   Solution: $V(x) = x(16 - 2x)(12 - 2x)$

   (b) Use the physical constraints to find the domain of V.
   Solution: $0 \leq x \leq 6$.

   (c) Evaluate V at the $x = 1, x = 2$, and $x = 3$.
   Solution: $V(1) = 1 \cdot 14 \cdot 10 = 140$, $V(2) = 2 \cdot 12 \cdot 8 = 192$, and $V(3) = 3 \cdot 10 \cdot 6 = 180$.

   (d) Find the derivative of V and use it to find the places where the tangent line is horizontal.
   Solution: Rewrite $V(x)$ as $V(x) = (16x - 2x^2)(12 - 2x)$ and use the product rule to get $V'(x) =$

$$\begin{aligned} (16 - 4x)(12 - 2x) + 16x - 2x^2)(-2) &= \\ 8x^2 - 80x + 12 \cdot 16 + 4x^2 - 32x &= \\ 12x^2 - 112x + 12 \cdot 16 &= \\ 4(3x^2 - 28x + 48). \end{aligned}$$

   Alternatively, $V(x) = 4x^3 - 56x^2 + 192x$, so $V'(x) = 12x^2 - 112x + 192 = 4(3x^2 - 28x + 48)$.

   (e) Find the critical points of V (ie, the places where the tangent line is horizontal) and pick out the one that belongs to the domain of V. Estimate this critical point to the nearest tenth of a unit. Estimate the value of V at that point.
   Solution: Using the quadratic formula,

$$x = \frac{28 \pm \sqrt{28^2 - 4 \cdot 3 \cdot 48}}{6}.$$

   The value in the domain is $x \approx 2.26$. And $V(2.26) \approx 194.07$

5. (12 points) Find the domain of the function

$$g(x) = \sqrt{x(x+1)(x-1)(x-3)}.$$

Express your answer as a union of intervals. That is, use interval notation.

Solution: The function is defined for those x for which $x(x+1)(x-1)(x-3) \geq 0$, that is $(-\infty, -1] \cup [0, 1] \cup [3, \infty)$.

6. (12 points) Let $H(x) = (x^2-4)^2(x-3)^2$. Using the chain rule and the product rule,

$$H'(x) = 2(x^2-4) \cdot 2x(x-3)^2 + (x^2-4)^2 \cdot 2(x-3).$$

Three of the zeros of $H'(x)$ are $x = \pm 2$ and $x = 3$. Find the other two.

Solution: Factor out the common terms to get $H'(x) = 2(x^2-4)(x-3)[2x(x-3)+(x^2-4)]$. One factor is $2x^2 - 6x + x^2 - 4 = 3x^2 - 6x - 4$. Apply the quadratic formula to get $x = \frac{6 \pm \sqrt{36+4 \cdot 4(3)}}{6}$ which reduces to $x = 1 \pm \frac{\sqrt{21}}{3}$.

7. (10 points) The demand curve for a new phone is given by $3p+2x=18$ where p is the price in hundreds of dollars and x is the number demanded in millions. The supply curve is given by $x-p^2+4p=3$. Find the point of equilibrium.

   Solution: Since $x=-3p/2+9$ and $x=p^2-4p+3$, we can solve $-3p/2+9=p^2-4p+3$ for p: $p^2-4p+3p/2+3-9=0$, so $p^2-5p/2-6=0$, so $2p^2-5p-12=0$, which can be solved by factoring. $(2p+3)(p-4)=0$, which has $p=4$ and therefore $x=3$ as a solution.

8. (10 points) Suppose $p(x)$ is a polynomial of degree 5 and $q(x)$ is a polynomial of degree 6. What is the degree of the polynomial $H(x)=(x^2p(x)-1)^2-(q(x)+x^2)^2+x^{13}$? Write a sentence about your reasoning.

   Solution: We can reason as follows: $H(x)$ is the sum of three polynomials $(x^2p(x)-1)^2, -(q(x)+x^2)^2$ and x^{13}, and their degrees are, respectively $14, 12$ and 13, so H has degree 14.

9. (15 points) Let

$$f(x) = \begin{cases} |x-3| & \text{if } x < 2 \\ 1 & \text{if } x = 2 \\ x - 2 & \text{if } 2 < x \leq 4 \\ x^2 - 14 & \text{if } 4 < x \end{cases}$$

(a) What is $\lim_{x\to 2^-} f(x)$?
Solution: $\lim_{x\to 2^-} f(x) = 1$

(b) What is $\lim_{x\to 2^+} f(x)$?
Solution: $\lim_{x\to 2^+} f(x) = 0$

(c) What is $\lim_{x\to 2} f(x)$?
Solution: $\lim_{x\to 2} f(x)$ does not exist.

(d) What is $\lim_{x\to 4^-} f(x)$?
Solution: $\lim_{x\to 4^-} f(x) = 2$

(e) What is $\lim_{x\to 4^+} f(x)$?
Solution: $\lim_{x\to 4^+} f(x) = 2$

(f) What is $\lim_{x\to 4} f(x)$?
Solution: $\lim_{x\to 4} f(x) = 2$

(g) What is $f(2)$?
Solution: $f(2) = 1$.

(h) What is $f(4)$?
Solution: $f(4) = 2$.

10. (25 points) Let $f(x) = \sqrt{2x+1}$. Notice that $f(4) = \sqrt{2 \cdot 4 + 1} = 3$.

(a) Find the slope of the line joining the two points $(4, f(4))$ and $(5, f(5))$.

Solution: The slope is $\frac{f(5)-f(4)}{5-4} = \frac{\sqrt{11}-3}{1} \approx 0.317$.

(b) Let h be a positive number. What is the slope of the line passing through the points $(4, f(4)$ and $(4+h, f(4+h)$. Your answer depends on h, of course.

Solution: $\frac{f(4+h)-f(4)}{h} = \frac{\sqrt{2(4+h)+1}-\sqrt{2 \cdot 4+1}}{h}$.

(c) Compute $\lim_{h\to 0} \frac{f(4+h)-f(4)}{h}$ to get $f'(4)$.

Solution: Since we get zero over zero, we recall that, in this case, we should rationalize the numerator.

$$
\begin{aligned}
\lim_{h\to 0} \frac{f(4+h)-f(4)}{h} &= \lim_{h\to 0} \frac{\sqrt{2(4+h)+1}-\sqrt{2\cdot 4+1}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2(4+h)+1}-\sqrt{9}}{h} \cdot \frac{\sqrt{8+2h+1}+3}{\sqrt{8+2h+1}+3} \\
&= \lim_{h\to 0} \frac{9+2h-9}{h(\sqrt{2(4+h)+1}+3)} \\
&= \lim_{h\to 0} \frac{2h}{h(\sqrt{2(4+h)+1}+3)} \\
&= \lim_{h\to 0} \frac{2}{(\sqrt{2(4+h)+1}+3)} \\
&= \frac{2}{2(3)} \\
&= \frac{1}{3}
\end{aligned}
$$

So, $f'(4) = 1/3$.

(d) Your answer to (c) is the slope of the line tangent to the graph of f at the point $(4, f(4))$. In other words, your answer is $f'(4)$. Write and equation for the tangent line.

Solution: The line is $y - 3 = \frac{1}{3}(x-4)$, or $y = x/3 + 5/3$.