March 2, 2006 **Name** ______________________________

The total number of points available is 138. Throughout this test, **show your work.**

1. (12 points) Let $f(x) = \sqrt{x^3 - x + 3}$.

   (a) Compute $f'(x)$

   Solution: $f'(x) = \frac{1}{2}(x^3 - x + 3)^{-1/2} \cdot 3x^2 - 1 = \frac{3x^2-1}{2\sqrt{x^3-x+3}}$.

   (b) What is $f'(2)$?

   Solution: $f'(2) = \frac{3 \cdot 2^2 - 1}{2\sqrt{2^3-2+3}} = 11/6$

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(2, f(2))$.

   Solution: Since $f(2) = 3$, using the point-slope form leads to $y - 3 = f'(2)(x - 2) = 11(x - 2)/6$, so $y = 11x/6 - 2/3$.

2. (12 points) Consider the function f defined by:

$$f(x) = \begin{cases} 3x - x^3 & \text{if } x < 1 \\ 3 & \text{if } x = 1 \\ 2x^{2/3} & \text{if } x > 1 \end{cases}$$

   (a) Is f continuous at $x = 1$?

   Solution: No, the limits from the left and right are both 2, but the value of f at 1 is 3.

   (b) What is the slope of the line tangent to the graph of f at the point $(8, 8)$?

   Solution: To find $f'(8)$ first note that when x is near 8, $f(x) = 2x^{2/3}$ so $f'(x) = 2\frac{2}{3}x^{-1/3}$. Thus, $f'(8) = 2\frac{2}{3}8^{-1/3} = 2\frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}$.

   (c) Find $f'(-3)$

   Solution: To find $f'(-3)$, we must differentiate the part of f defined for $x < 1$. In this area, $f'(x) = 3 - 3x^2$, so $f'(-3) = 3 - 3(-3)^2 = -24$.

3. (12 points) If a ball is thrown vertically upward from the roof of 112 foot building with a velocity of 48 ft/sec, its height after t seconds is $s(t) = 112 + 48t - 16t^2$.

   (a) What is the height the ball at time $t = 0$?

   Solution: $s(0) = 112$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   Solution: $s'(t) = 0$ when the ball reaches its max height.

   (c) What is the maximum height the ball reaches?

   Solution: Solve $s'(t) = 48 - 32t = 0$ to get $t = 3/2$ when the ball reaches its zenith. Thus, the max height is $s(3/2) = 112 + 48(3/2) - 16(3/2)^2 = 148$.

   (d) What is the velocity of the ball when it hits the ground (height 0)?

   Solution: Solve $s(t) = 0$ using the quadratic formula to get $t = \frac{3 \pm \sqrt{9+28}}{2} = \frac{3 \pm \sqrt{37}}{2}$, but the larger is only reasonable answer. Find $s'((3+\sqrt{37})/2) \approx -97.3$ feet/sec.

4. (7 points) The cost of producing x units of stuffed alligator toys is $C(x) = 0.003x^2 + 6x + 6000$. Find the marginal cost at the production level of 1000 units.

   Solution: $C'(x) = \frac{d}{dx} 0.003x^2 + 6x + 6000 = 0.006x + 6$ so $C'(1000) = 6 + 6 = 12$.

5. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = f(x) \cdot g(x)$. Compute $L'(2)$.

Solution: $L'(x) = (g'(x)f(x) + f'(x)g(x))$, so $L'(2) = (f(2)g'(2) + g(2)f'(2)) = (6 \cdot 4 + 4 \cdot 3) = 36$.

(b) Let $U(x) = g \circ g(x)$. Compute $U'(1)$.

Solution: By the chain rule, $U'(x) = g'(g(x)) \cdot g'(x)$, so $U'(1) = g'(g(1)) \cdot g'(1) = g'(2) \cdot g'(1) = 4 \cdot 5 = 20$.

(c) Let $K(x) = g(x^2) \cdot f(x)$. Compute $K'(1)$

Solution: $K'(x) = g'(x^2) \cdot 2x \cdot f(x) + f'(x)g(x^2)$, so $K'(1) = g'(1) \cdot 2 \cdot f(1) + f'(1)g(1) = 5 \cdot 2 \cdot 4 + 6 \cdot 2 = 52$.

(d) Let $V(x) = f(g(2x))$. Compute $V'(3)$.

Solution: Again, by the chain rule, $V'(x) = f'(g(2x)) \cdot g'(2x) \cdot 2$, so $V'(5) = f'(g(6)) \cdot g'(6) \cdot 2 = f'(2) \cdot g'(6) \cdot 2 = 4 \cdot 4 \cdot 2 = 32$.

(e) Let $W(x) = [g(2x - f(x))]^2$. Compute $W'(4)$.

Solution: Again by the chain rule, $W'(x) = 2g(2x - f(x))^1 \cdot g'(2x - f(x)) \cdot (2 - f'(x))$, so $W'(4) = 2g(8 - f(4)) \cdot g'(8 - f(4)) \cdot (2 - f'(4)) = 2g(5) \cdot g'(5) \cdot (2 - f'(4)) = 2 \cdot 4 \cdot 1(2 - 5) = -24$.

(f) Let $Z(x) = g(x^2 + f(x))$. Compute $Z'(1)$.

Solution: Again by the chain rule and the product rule, $Z'(x) = g'(x^2 + f(x)) \cdot \frac{d}{dx}(x^2 + f(x)) = g'(x^2 + f(x)) \cdot (2x + f'(x))$, so $Z'(3) = g'(1 + f(1)) \cdot (2 + f'(1)) = g'(5) \cdot (2 + 6) = 1 \cdot 8 = 8$.

6. (25 points) Compute the following derivatives.

   (a) Let $f(x) = (x + \sqrt{1 + x^3})^2$. Find $\frac{d}{dx} f(x)$.
   Solution: Note that $\sqrt{x^3} = x^{3/2}$, so we differentiate it using the power rule and chain rule: $f'(x) = 2(x + \sqrt{1 + x^3}) \cdot (1 + \frac{1}{2}(1 + x^3)^{-1/2} \cdot 3x^2)$.

   (b) Let $g(x) = x^3/(1 + x^2)$. What is $g'(x)$?
   Solution: Use the quotient rule to get $g'(x) = 3x^2(1 + x^2) - 2x(x^3) \div (1 + x^2)^2 = \dfrac{x^4 + 3x^2}{(1 + x^2)^2}$.

   (c) Find $\frac{d}{dx}((x + 2)^2 \cdot (2x - 1))$.
   Solution: By the product rule, $\frac{d}{dx}((x + 2)^2 \cdot (2x^3 - 1)) = 2(x + 2) \cdot (2x - 1) + 2(x + 2)^2 = 2(x + 2)(3x + 1)$.

   (d) Find $\frac{d}{dx} \sqrt{\frac{2x^3+1}{3x-2}}$.
   Solution: By the chain and quotient rules, $\frac{d}{dx} \dfrac{2x^3 + 1}{x - 2} = \dfrac{1}{2} \left(\dfrac{2x^3 + 1}{3x - 2} \right)^{-1/2} \cdot \dfrac{6x^2(3x - 2) - 3(2x^3 + 1)}{(3x - 2)^2}$.

   (e) Find $\frac{d}{dt}(t^2 + 1/t^2)^4$.
   Solution: By the chain rule, $\frac{d}{dt}(t^3 + 1/t)^4 = 4(t^3 + 1/t)^3 \cdot (3t^2 - t^{-2})$.

7. (40 points) Consider the rational function

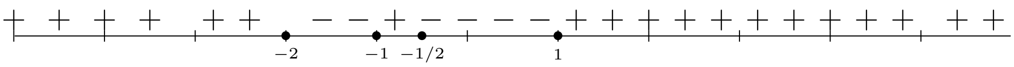
$$r(x) = \frac{(x^2-4)(2x+1)}{(3x^2-3)(x-2)}.$$

Use the Test Interval Technique to solve the inequality $r(x) \geq 0$.

Solution: Notice first that f is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$r(x) = \frac{(x-2)(x+2)(2x+1)}{3(x-1)(x+1)(x-2)}.$$

We can cancel the common factors with the understanding that we are (very slightly) enlarging the domain of r: $r(x) = \frac{(x+2)(2x+1)}{3(x-1)(x+1)}$. Next find the branch points. These are the points at which f can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $-2, -1/2, 1, -1$. Again we select test points and find the sign of f at of these points to get the sign chart.



Again suppose that we are solving $f(x) \geq 0$ The solution to $f(x) > 0$ is easy. It is the union of the open intervals with the + signs, $(-\infty, -2) \cup (-1, -1/2) \cup (1, \infty)$. It remains to solve $f(x) = 0$ and attach these solutions to what we have. The zeros of f are -2 and $-1/2$. So the solution to $f(x) \geq 0$ is $(-\infty, -2] \cup (-1, -1/2] \cup (1, \infty)$. Notice that the branch points 1 and -1 are not included since f is not defined at these two points. It has vertical asymptotes at these two places. Technically the value $x = 2$ should not be included in the solution because the function f as originally defined is not defined at $x = 2$. Thus, the exact answer is $(-\infty, -2] \cup (-1, -1/2] \cup (1, 2) \cup (2, \infty)$.