March 13, 2015 **Name** ______________________________

The problems count as marked. The total number of points available is 146. Throughout this test, **show your work.** Using a calculator to circumvent ideas discussed in class will generally result in no credit. Note please that this test is a composite of the tests for sections 1 and 2.

1. (24 points) Let
$$H(x) = 3x^4 + 4x^3 - 72x^2.$$

   (a) Find $H'(x)$ and $H''(x)$.
   Solution: $H'(x) = 12x^3 + 12x^2 - 144x$, and $H''(x) = 36x^2 + 24x - 144$.

   (b) Find all the critical points of H.
   Solution: $H'(x) = 0$ at $x = 0, x = 3$ and $x = -4$.

   (c) Find the intervals over which $H(x)$ is increasing.
   Solution: Building the sign chart for $H'(x)$, we see that H is increasing precisely over the intervals $(-4, 0)$ and $(3, \infty)$.

   (d) Discuss the concavity of H.
   Solution: H is concave downwards between the roots of $H''(x) = 0$, $x = \frac{-2\pm\sqrt{148}}{6}$, and concave upward elsewhere.

2. (12 points) Consider the function f defined by:
$$f(x) = \begin{cases} 3x - x^3 & \text{if } -4 < x < 1 \\ 2x^{2/3} & \text{if } 1 < x < 10 \end{cases}$$

   (a) What is the domain of f? Write your answer in interval notation.
   Solution: $(-4, 1) \cup (1, 10)$.

   (b) What is the slope of the line tangent to the graph of f at the point $x = 8$?
   Solution: To find $f'(8)$ first note that when x is near 8, $f(x) = 2x^{2/3}$ so $f'(x) = 2\frac{2}{3}x^{-1/3}$. Thus, $f'(8) = 2\frac{2}{3}8^{-1/3} = 2\frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}$.

   (c) Find an equation for th line tangent to the graph of f at $(-3, f(-3))$.
   Solution: To find $f'(-3)$, we must differentiate the part of f defined for $x < 1$. In this area, $f'(x) = 3 - 3x^2$, so $f'(-3) = 3 - 3(-3)^2 = -24$. So the line we seek is $y - 18 = -24(x + 3)$

3. (15 points) Let $f(x) = \sqrt{x^4 - 3x + 11}$.

   (a) Compute $f'(x)$
   Solution: $f'(x) = \frac{1}{2}(x^4 - 3x + 11)^{-1/2}(4x^3 - 3) = \frac{4x^3-3}{2\sqrt{x^4-3x+11}}$.

(b) What is $f'(1)$?

Solution: $f'(1) = \frac{4-3}{2\sqrt{1-3+11}} = 1/6$

(c) Use the information in (b) to find an equation for the line tangent to the graph of f and at the same time perpendicular to the line $y - 3 = -6(x + 4)$.

Solution: Since $f(1) = 3$, using the point-slope form leads to $y - 3 = f'(1)(x - 1) = (x - 1)/6$, so $y = x/6 + 17/6$.

4. (20 points)

   (a) Use the product rule for two functions (that you learned in class) to build a product rule for three functions. In other words, suppose $H(x) = f(x) \cdot g(x) \cdot h(x)$. Build a formula for $H'(x)$ in terms of the derivatives of f, g and h.

   Solution: Use the product rule twice to write $H'(x) = f'(x)[g(x) \cdot h(x)] + f(x)[g'(x)h(x) + h'(x)g(x)] = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x)$.

   (b) Use your rule to find $\frac{d}{dx}(2x-3)(3x+1)(5x-3)$. You must show that you used your 'new' product rule to get credit in this problem. Of course you can differentiate the cubic function to check your answer.

   Solution: Let $H(x) = (2x-3)(3x+1)(5x-3)$. Then $H'(x) = 2(3x+1)(5x-3) + 3(2x-3)(5x-3) + 5(2x-3)(3x+1)$.

   (c) How many critical points does H have? Discuss your reasoning.

   Solution: The derivative is a quadratic polynomial so it can have at most two zeros. Since the discriminant $b^2 - 4ac = 106^2 - 4(90)(6)$ is positive, it has two zeros.

5. (20 points) Let
$$H(x) = (2x+7)^2(x^2-9).$$

   (a) Find $H'(x)$ using the product and chain rules.
   Solution: $H'(x) = 2(2x+7)\cdot 2(x^2-9) + 2x(2x+7)^2 = 2(2x+7)[2x^2 - 18 + 2x^2 + 7x] = 2(2x+7)(4x^2+7x-18)$.

   (b) Find all the critical points of H. Its fine to leave these is radical form.
   Solution: Obviously, $x = -7/2$ is one critical point. The other two are the zeros of the quadratic $4x^2+7x-18$. These are $\alpha = \frac{-7-\sqrt{49+288}}{8}$ and $\beta = \frac{-7+\sqrt{49+288}}{8}$. So $\alpha \approx -3.17$ and $\beta \approx 1.42$.

   (c) Find the intervals over which $H(x)$ is increasing.
   Solution: Building the sign chart for $H'(x)$, we see that H is increasing precisely over the intervals $(-7/2, \alpha)$ and (β, ∞).

6. (20 points) Let $f(x) = x^4/4 - x^2 - x$. Follow the instructions below to prove that f has exactly one critical point in the interval $[1, 2]$. You may assume that all polynomials are continuous for all real numbers. Thus the hypothesis of IVT is satisfied.

   (a) Use the Intermediate Value Theorem to prove that f has at least one critical point in $[1, 2]$.

   Solution: Note first that $f'(x) = x^3 - 2x - 1$. Next $f'(1) = -2$ and $f'(2) = 3$, so we can invoke IVT to guarantee a zero between 1 and 2.

   (b) Use the Big Theorem we discussed in class (about finding intervals where a function increases) to prove that f can have at most 1.

   Solution: First note that $f''(x) = 3x^2 - 2$ which has two zeros, both less than 1. Since $f''(x) > 0$ on $[1, 2]$, it follows that $f'(x)$ is increasing throughout the interval and so it has at most one zero.

7. (20 points) Build a cubic polynomial $p(x)$ that has a relative maximum at $x = -2$ and a relative minimum at $x = 3$.

   Solution: The derivative of $p(x)$ must have zeros at $x = -2$ and $x = 3$, so we can guess to write $p'(x) = (x + 2)(x - 3) = x^2 - x + 6$. So $p(x)$ could be a multiple of $x^3/3 - x^2/2 + 6x$. One nice multiple is $p(x) = 2x^3 - 3x^2 + 36x$, which we can analyze to see that it works.

8. (35 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = \sqrt{x} + f(x) + g(x)$. Compute $L(4)$ and $L'(4)$.

Solution: $L(4) = \sqrt{4} + f(4) + g(4) = 2 + 3 + 2 = 7$ and $L'(x) = \frac{1}{2}x^{-1/2} + f'(x) + g'(x)$, so $L'(4) = \frac{1}{2}4^{-1/2} + f'(4) + g'(4) = 11.25$.

(b) Let $U(x) = x^2 f(x)$. Compute $U(3)$ and $U'(3)$.

Solution: First, note that $U(3) = 9f(3) = 9$. By the product rule, $U'(x) = 2xf(x) + x^2 f'(x)$, so $U'(3) = 2 \cdot 3 \cdot f(3) + 9 \cdot f'(3) = 6 + 18 = 24$.

(c) Let $K(x) = f(x^2)/x$. Compute $K(2)$ and $K'(2)$

Solution: First, $K(2) = f(4)/2 = 3/2$. Next $K'(x) = [f'(x^2)2x \cdot x - f(x^2)] \div x^2$, so $K'(2) = [f'(4) \cdot 8 - f(2^2)] \div 4 = [40 - 3]/4 = 9.25$.

(d) Let $Z(x) = (2x + g(x))^3$. Compute $Z(1)$ and $Z'(1)$.

Solution: First $Z(1) = (2 + g(1))^3 = 64$. By the chain rule, $Z'(x) = 3(2x + g(x))^2[2 + g'(x)]$ so $Z'(1) = 3 \cdot 4^2[2 + 5] = 336$.

(e) Let $Q(x) = g(x^2 + f(2x))$. Compute $Q(0)$ and $Q'(0)$.

Solution: First, $Q(0) = g(0 + f(0)) = g(2) = 3$. Again by chain rule, $Q'(x) = g'(x^2 + f(x)) \cdot (2x + f'(2x) \cdot 2$ so $Q'(0) = g'(f(0))(f'(0) \cdot 2 = 4 \cdot 1 \cdot 2 = 8$.

9. (24 points) Consider the polynomial $g(x) = 2x^3 + 3x^2 - 36x + 10$.

   (a) Find the two critical points of g.

   Solution: Since $g'(x) = 6x^2 + 6x - 36$, it follows that $x = 2$ and $x = -3$ are critical points.

   (b) Build the sign chart for the function $g'(x)$.

   Solution: $g'(x) < 0$ precisely on $(-3, 2)$.

   (c) Classify the critical points of g as (a) relative maxima, (b) relative minima, or (c) imposters.

   Solution: $x = -3$ is the location of a relative maximum, and $x = 2$ is the place where a relative minimum occurs.

   (d) Find an interval where g is concave upwards.

   Solution: Since $g''(x) > 0$ if $x > -1/2$, it follows that g is concave upwards on $(-1/2, \infty)$.

   (e) Find a point of the graph of g where the tangent line is parallel to the line $y = -36x - 4$.

   Solution: The equation $g'(x) = 6x^2 + 6x - 36 = -36$ has two solutions, $x = 0$ and $x = -1$.