June 13, 2001 **Name** ______________________________

The total number of points possible is 130. **SHOW YOUR WORK**

1. (20 points) Use the definition of derivative to find $f'(a)$ for the function $f(x) = 4x - x^3$. Use this information to find the slope of the line tangent to the graph of f at the point $(-1, -3)$.

   Solution:

$$\begin{aligned}
\lim_{h\to 0} \frac{f(x+h) - f(x)}{h} &= \\
\lim_{h\to 0} \frac{4(x+h) - (x+h)^3 - (4x - x^3)}{h} &= \\
\lim_{h\to 0} \frac{4x + 4h - (x^3 + 3x^2h + 3xh^2 + h^3) - 4x + x^3}{h} &= \\
\lim_{h\to 0} \frac{4h - 3x^2h - 3xh^2 - h^3}{h} &= \\
\lim_{h\to 0} \frac{h(4 - 3x^2 - 3xh - h^2)}{h} &= \\
\lim_{h\to 0} 4 - 3x^2 - 3xh - h^2 &= 4 - 3x^2,
\end{aligned}$$

   so the slope of the tangent line at $(-1, -3)$ is $f'(-1) = 4 - 3(-1)^2 = 1$.

2. (10 points) Find the derivative of $f(x) = (2x^2 - \sqrt{x})^2$.

   Solution: By the chain rule, $f'(x) = 2(2x^2 - \sqrt{x})(4x - \frac{1}{2}x^{-1/2})$.

3. (10 points) Find $\frac{dy}{dx}$ when $y = (x^2 - 7x + 1)(3x - 1/x)$

   Solution: By the product rule, $\frac{dy}{dx} = (2x-7)(3x-1/x)+(3+x^{-2})(x^2-7x+1)$.

4. (10 points) Find an equation for the line tangent to the graph of $h(x) = \dfrac{3x-2}{x^2-1}$ at the point $(0, 2)$.

   Solution: By the quotient rule, $h'(x) = \frac{3(x^2-1)-2x(3x-2)}{(x^2-1)^2}$, so $h'(0) = \frac{3(-1)-2\cdot 0(3\cdot 0-2)}{1} = -3$, so the tangent line has equation $y - 2 = -3(x - 0)$ which simplifies to $y = -3x + 2$.

   Solution: By the quotient rule,

$$\begin{aligned} h'(x) &= \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2} \\ &= \frac{3(x^2-1) - 2x(3x-2)}{(x^2-1)^2} \\ &= \frac{3x^2 - 3 - 6x^2 + 4x}{(x^2-1)^2} \\ &= \frac{-3x^2 + 4x - 3}{(x^2-1)^2}. \end{aligned}$$

   Thus $h'(0) = -3/1 = -3$ and the tangent line is given by $y - 2 = -3x$ or $y = -3x + 2$.

5. (10 points) The total weekly cost in dollars incurred by the Lincoln Record Company in pressing x playing records is given by $C(x) = 3000 + 3x - 0.001x^2,\ 0 \le x \le 6000$.

   (a) Find the average cost function $\overline{C}$.
   Solution: $\overline{C} = \frac{C(x)}{x} = \frac{3000+3x-0.001x^2}{x} = 3000x^{-1} + 3 - 0.001x$.

   (b) Find the marginal average cost function $\overline{C'}$.
   Solution: $\overline{C'} = -3000x^{-2} - 0.001$.

6. (10 points) Does the function $f(x) = \sqrt{x+3}$ satisfy the hypothesis of Intermediate Value Theorem over the interval $[-2, 6]$. If so, find an INTEGER (ie, a whole number) M between $f(-2)$ and $f(6)$, and then find a number c in the interval $(-2, 6)$ such that $f(c) = M$.

   Solution: The only integer between $f(-2) = \sqrt{-2+3} = 1$ and $f(6) = \sqrt{6+3} = 3$ is 2 , so $M = 2$. We need to solve $f(c) = \sqrt{c+3} = 2$. Squaring both sides yields $c + 3 = 4$, and it follows that $c = 1$.

7. (10 points) Suppose that $f'(3) = 2$ and $f(3) = 1$. What is the y-intercept of the line tangent to the graph of f at the point $(3, 1)$?

   Solution: $y - 1 = 2(x - 3)$ is equivalent to $y = 2x - 5$ so the y-intercept is -5.

8. (30 points) Suppose the functions f and g are differentiable. Some of the values of f, f', g, and g' are given in the table. The next six problems refer to these functions f and g. Recall that, for example, the entry 10 in the fifth row and sixth column means that $g'(4) = 10$.

x	$f(x)$	$f'(x)$
0	2	1
1	7	3
2	5	4
3	1	2
4	3	3
5	6	4
6	0	5
7	4	1

x	$g(x)$	$g'(x)$
0	5	5
1	7	3
2	4	4
3	2	6
4	6	10
5	3	4
6	1	2
7	0	1

   (a) The function h is defined by $h(x) = f(g(x))$. Use the chain rule to find $h'(3)$.

   Solution: Since $h'(x) = f'(g(x)) \cdot g'(x)$ for all x, $h'(3) = f'(g(3) \cdot g'(3) = f'(2)g'(3) = 4 \cdot 6 = 24$.

   (b) The function R is defined by $R(x) = g(f(x))$. Use the chain rule to find $R'(2)$.

   Solution: Since $R'(x) = g'(f(x)) \cdot f'(x)$ for all x, $R'(3) = g'(f(2) \cdot f'(2) = g'(5)f'(2) = 4 \cdot 4 = 16$.

   (c) The function k is defined by $k(x) = f(x) \cdot g(x)$. Use the product rule to find $k'(5)$.

   Solution: Since $k'(x) = f'(x)g(x) + g'(x)f(x)$, $k'(5) = f'(5)g(5) + g'(5)f(5) = 36$.

   (d) The function H is defined by $H(x) = f(x)/g(x)$. Use the quotient rule to find $H'(4)$.

   Solution: Since $H'(x) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x)^2}$ for all x, $H'(4) = \frac{-4 \cdot 3}{36} = -\frac{1}{3}$.

   (e) The function K is defined by $K(x) = (f(x) + g(x))^2$. Find $K'(6)$.

   Solution: By the chain rule, $K'(x) = 2(f(x) + g(x)) \cdot (f'(x) + g'(x))$ for all x, $K'(6) = 2(f(6) + g(6)) \cdot (f'(6) + g'(6)) = 2(0 + 1)(5 + 2) = 14$

(f) The function M is defined by $M(x) = f(f(x))$. Use the chain rule to find $M'(0)$.
Solution: By the chain rule, $M'(x) = f'(f(x) \cdot f'(x)$ for all x. Thus, $M'(0) = f'(f(0)) \cdot f'(0) = f'(2) \cdot f'(0) = 4 \cdot 1 = 4$.

9. (20 points) The altitude of a rocket t seconds into flight is given

$$s = f(t) = -2t^3 + 114t^2 + 480t + 1 \qquad (t \geq 0),$$

where s is measured in feet.

(a) Find an expression v for the rockets velocity at any time t.
Solution: $v(t) = s'(t) = -6t^2 + 228t + 480$.

(b) Compute the rockets velocity when $t = 10, 40, 50$, and 70. Interpret your results.
Solution: $v(10) = 2160$, $v(40) = 0$, $v(50) = -3120$, and $v(70) = -12960$.

(c) Using the results from part b., find the maximum height of the rocket. Hint: at its maximum height, the velocity of the rocket is zero.
Solution: Since $v(40) = 0$, it follows that the maximum height is attained at $t = 40$ seconds. The position of the rocket after 40 seconds is $s(40) = -2 \cdot 40^3 + 114 \cdot 40^2 + 480 \cdot 40 + 1 = 73601$ feet.