November 1, 2017 **Name** ___________ The total number of points available is 152. Throughout this test, **show your work.**

1. (10 points) Let $f(x) = x^3 - 2x - 3$.

   (a) Compute $f'(x)$
   Solution: $f'(x) = 3x^2 - 2$

   (b) What is $f'(2)$?
   Solution: $f'(2) = 3 \cdot 2^2 - 2 = 10$.

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(2, f(2))$.
   Solution: Since $f(2) = 2^3 - 2 \cdot 2 - 3 = 1$, using the point-slope form leads to $y - 1 = f'(2)(x - 2) = 10(x - 2)$, so $y = 10x - 19$.

2. (12 points) Consider the function f defined by:

$$f(x) = \begin{cases} x + x^3 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 2x^{1/2} & \text{if } x > 1 \end{cases}$$

   (a) Is f continuous at $x = 1$?
   Solution: Yes, the limits from the left and right are both 2, and the value of f at 1 is 2.

   (b) What is the slope of the line tangent to the graph of f at the point $(4, 4)$?
   Solution: To find $f'(4)$ first note that when x is near 8, $f(x) = 2x^{1/2}$ so $f'(x) = 2\frac{1}{2}x^{-1/2}$. Thus, $f'(4) = 2 \cdot \frac{1}{2} \cdot 4^{-1/2} = \frac{1}{2} = $.

   (c) Find $f'(-3)$
   Solution: To find $f'(-3)$, we must differentiate the part of f defined for $x < 1$. In this area, $f'(x) = 1 + 3x^2$, so $f'(-3) = 1 + 3(-3)^2 = 28$.

3. (25 points) If a stone is shot vertically upward from the roof of 212 foot building with a velocity of 320 ft/sec, its height after t seconds is $s(t) = 212+320t-16t^2$.

   (a) What is the height the stone at time $t = 0$?
   Solution: $s(0) = 212$.

   (b) What is the height the stone at time $t = 2$?
   Solution: $s(2) = 788$.

   (c) What is the average velocity of the stone during the third second?
   Solution: $S(3) = 1028$ and $S(2) = 788$ so we have $(1028 - 788)/1 = 240$

   (d) What is the average velocity of the stone during time interval $[2, 2.1]$?
   Solution: $\frac{S(2.1)-S(2.0)}{2.1-2.0} = \frac{212+320(2.1)-16(2.1)^2-212-320(2.0)+16(2.0)^2}{0.1} = \frac{320(2.1-2.0)-16(2.1)^2-2.0^2}{0.1} = \frac{32-16(.1)(4.1)}{0.1} = 10 \cdot 25.44 = 254.4\ ft/sec.$

   (e) What is the average velocity of the stone during time interval $[2, 2.01]$?
   Solution:
   Solution: $\frac{S(2.01)-S(2.0)}{2.01-2.0} = \frac{212+320(2.01)-16(2.1)^2-212-320(2.0)+16(2.0)^2}{0.01} = \frac{320(2.01-2.0)-16(2.01)^2-2.0^2}{0.01} = \frac{3.2-16(.01)(4.01)}{0.01} = 100 \cdot 2.558 = 255.8\ ft/sec.$

   (f) What is $s'(2)$?
   Solution: $V(2) = 320 - 16 \cdot 2^2 = 320 - 64 = 256$.

   (g) What is the velocity of the stone at the time it reaches its maximum height?
   Solution: $s'(t) = v(t) = 0$ when the stone reaches its max height.

   (h) At what time is the velocity zero?
   Solution: Solve $320 - 32t = 0$ to get $t = 10$.

   (i) What is the maximum height the stone reaches?
   Solution: Solve $s'(t) = 320 - 32t = 0$ to get $t = 10$ when the stone reaches its zenith. Thus, the max height is $s(10) = 212 + 320(10) - 16(10)^2 = 2123200 - 1600 = 1812$ft.

   (j) What is the velocity of the stone when it hits the ground (height 0)?
   Solution: Solve $s(t) = 0$ using the quadratic formula to get $t = \frac{20\pm\sqrt{400+53}}{2} = \frac{20\pm\sqrt{453}}{2} \approx 10 + 10.6 = 20.6$, since the larger value is the only reasonable answer. Find $s'(20.6) = 320 - 32(20.6) \approx -340.5$ feet/sec.

4. (20 points) Let $f(x) = (x^2 - 9)^{2/3}$. Note: some tests had the function $f(x) = (x^2 - 9)^{1/3}$ or similar variations. These two types of functions yield quite different answers.

   (a) What is the domain of f?

   Solution: f is defined for all real numbers.

   (b) Find all the critical points of f

   Solution: f has three critical points, two singular and one stationary. First, $f'(x) = 2x(x^2 - 9)^{-1/3} \div 3$. The zeros of this is $x = 0$. f' is undefined at $x = \pm 3$.

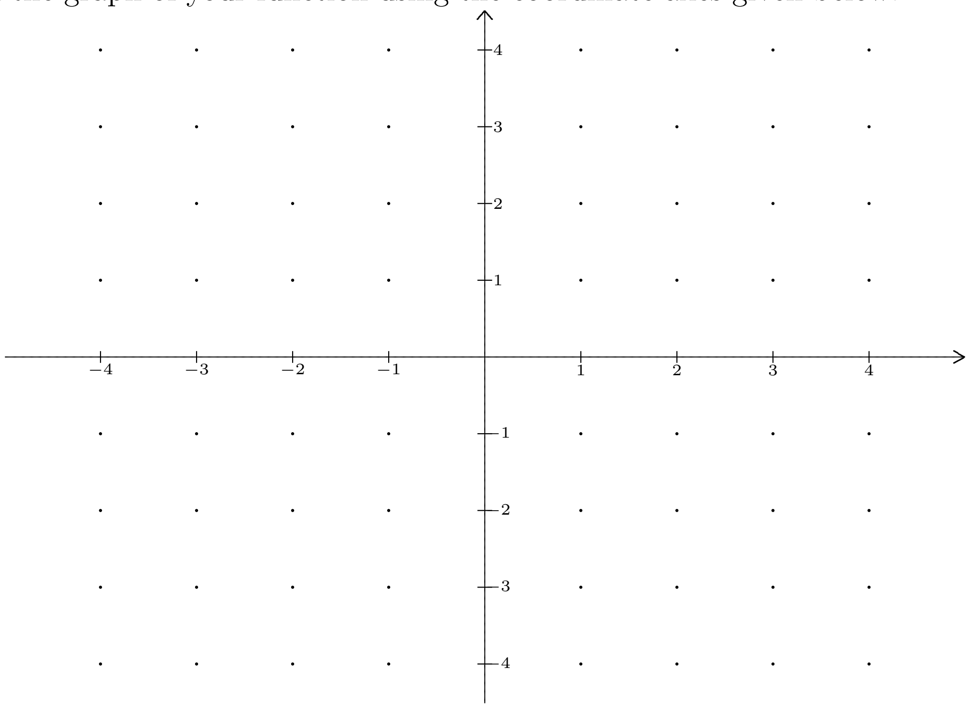
   (c) Identify each critical point of f as relative minimum, a relative maximum, or an imposter.

   Solution: The sign chart for f' shows sign changes at $-3, 0$ and -3. So f has relative mins at 3 and -3 and a rel. max $x = 0$.

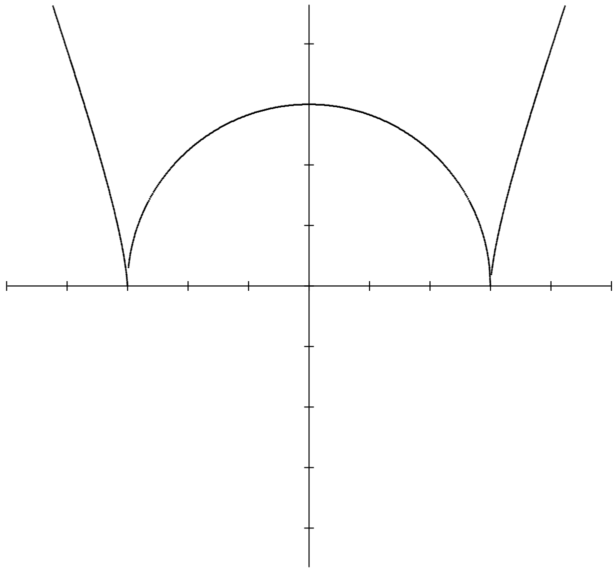
   (d) Build the sign chart for your function.

   Solution: Our r is positive on $(-\infty, -2) \cup (-1, 0) \cup (1, \infty)$.

   (e) Sketch the graph of your function using the coordinate axes given below.



   Solution: The following graph has the wrong value at zero. I hope to fix this soon.



5. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = f(x+1) + g(x-1)$. Compute $L(2)$ and $L'(2)$.
Solution: $L(2) = f(3) + g(1) = 3$ and $L'(x) = f'(x+1) + g'(x-1)$, so $L'(2) = f'(3) + g'(1) = 2 + 5 = 7$.

(b) Let $U(x) = g \circ f(x)$. Compute $U(1)$ and $U'(1)$.
Solution: $U(1) = g(f(1)) = g(4) = 2$. And $U'(x) = g'(f(x)) \cdot f'(x)$, so $U'(1) = g'(f(1) \cdot f'(1) = 6 \cdot 6 = 36$.

(c) Let $K(x) = g(x) \cdot f(x^2)$. Compute $K(2)$ and $K'(2)$.
Solution: $K(2) = g(2) \cdot f(4) = 3 \cdot 3 = 9$. and, by the product rule, $K'(x) = g'(x)f(x^2) + g(x) \cdot f'(x^2) \cdot 2x$. Therefore, $K'(2) = g'(2)f(4) + g(2)f'(4) \cdot 4 = 4 \cdot 3 + 3 \cdot 5 \cdot 4 = 72$.

(d) Again, $L(x) = g(x+2) \div f(2x-1)$. Compute $L(2)$ and $L'(2)$.
Solution: $L(2) = g(4) \div f(3) = 2/1 = 2$ and $L'(x) = (g'(x+2) \cdot f(2x-1) - 2f'(2x-1)g(x+2)) \div f(2x-1)^2$, so $L'(2) = g'(4) \cdot 4 \cdot f(2) + f'(2)g(4) = 6 \cdot 4 \cdot 6 + 4 \cdot 2 = 152$.

(e) Let $Z(x) = g(x^2 + f(x))$. Compute $Z(1)$ and $Z'(1)$.
Solution: $Z(1) = g(5) = 4$. Again by the chain rule, $Z'(x) = g'(x^2 + f(x)) \cdot \frac{d}{dx}(x^2 + f(x)) = g'(x^2 + f(x)) \cdot (2x + f'(x))$, so $Z'(1) = g'(1 + f(1)) \cdot (2 + f'(1)) = g'(3) \cdot (2 + 6) = 1 \cdot 8 = 8$.

6. (15 points) Two positive numbers x and y are related by $2x + 3y = 16$. What is the largest possible product xy could be, and what pair (x, y) achieves that product? Note that if $y = 2$, then $x = 5$ and the product $xy = 10$. If $y = 4$, then $x = 2$ and the product is 8. Trying various combinations of values is not worth any credit.

   Solution: Solve $2x + 3y = 16$ for y to get $f(x) = xy = x(\frac{16-2x}{3}) = \frac{16x-2x^2}{3}$. So $f'(x) = (16 - 4x)/3$ and $x = 4$ is the only critical point. So $x = 4$ and $y = 8/3$. It follows that the maximum value of xy is $4 \cdot 8/3 = 32/3$.

7. (10 points) The line tangent to the graph of a function f at the point $(2, 9)$ on the graph also goes through the point $(0, 7)$. What is $f'(2)$?

   Solution: The slope of the line through $(2, 9)$ and $(0, 7)$ is 1, so $f'(2) = 1$.

8. (30 points) Let $H(x) = (x^2 - 9)^2(3x + 1)^3$.

   (a) Use the chain and product rules to find $H'(x)$.

   Solution:

$$\begin{aligned} H'(x) &= 2(x^2-9)\cdot 2x(3x+1)^3 + 3(3x+1)^2 \cdot 3(x^2-9)^2 \\ &= (x^2-9)(3x+1)^2[4x(3x+1) + 9(x^2-9)] \\ &= (x-3)(x+3)(3x+1)^2[21x^2 + 4x - 81] \end{aligned}$$

   (b) Find the critical points of H.

   Solution: Thus, H has critical points $x = \pm 3, -1/3$ and $\frac{-4 \pm \sqrt{6820}}{42}$. The last two are roughly $\alpha \approx -2.06$ and $\beta \approx 1.87$.

   (c) Build the sign chart for $H'(x)$

   Solution: $H'(x) \geq 0$ on $(-\infty, -3] \cup [\alpha, \beta] \cup [3, \infty)$.

   (d) Classify the critical points of H as max, min, or imposters.

   Solution: Notice that $-1/3$ is an imposter since $h'(x)$ is positive on both sides of $-1/3$. Then we have maximums at -3 and β and minimums at α and 3.

   (e) Find the intervals over which H is increasing.

   Solution: From the sign chart for H', we see that H is increasing on $(-\infty, -3] \cup [\alpha, \beta] \cup [3, \infty)$.

9. (20 points) Let $f(x) = x^3 + x - 3$. Prove that f has exactly one zero as follows.

   (a) Use the Intermediate Value Theorem to show that f has at least one zero.

   Solution: First note that $f(1) = -1$ and $f(2) = 7$. Note that f is a polynomial, so its continuous. Thus we can apply the IVT to conclude that f has a zero in $[1, 2]$.

   (b) Prove that f is an increasing function on its domain. Conclude that f cannot have more than one zero.

   Solution: Take the derivative to get $f'(x) = 3x^2 + 1$ and note that f' is positive for all real x. Therefore f is an increasing function. Increasing functions cannot have more than 1 zero.