March 25, 2019 **Name** ____________________

The problems count as marked. The total number of points available is 156. Throughout this test, **SHOW YOUR WORK.**

1. (24 points) Demonstrate your understanding of the product, quotient and chain rules by differentiating each of the given functions. No need to simplify. You must show your work.

   (a) Let $F(x) = (x^2 - 3x + 1)(x^3 - 2x + 5)$

   Solution: Note that $F'(x) = (2x-3)(x^3-2x+5)+(3x^2-2)(x^2-3x+1)$.

   (b) $G(x) = \dfrac{2x^4 - 3x + 1}{x^2 - x + 3}$

   Solution: By the quotient rule, $G'(x) = \frac{(8x^3-3)(x^2-x+3)-(2x-1)(2x^4-3x+1)}{(x^2-x+3)^2}$.

   (c) $K(x) = (x^2 - 3)^{17}$

   Solution: By the chain rule, $K'(x) = 17(x^2 - 3)^{16} \cdot 2x = 34x(x^2 - 3)^{16}$.

   (d) $H(x) = \sqrt{(3x + 1)^4 - 7}$.

   Solution: By the chain rule, $H'(x) = \frac{1}{2}((3x+1)^4-7)^{-1/2} \cdot 4(3x+1)^3 \cdot 3 = \frac{6(3x+1)^3}{\sqrt{(3x+1)^4-7}}$.

2. (20 points) Consider the function $f(x) = \frac{(2x+3)(x-3)}{x(x-1)}$.

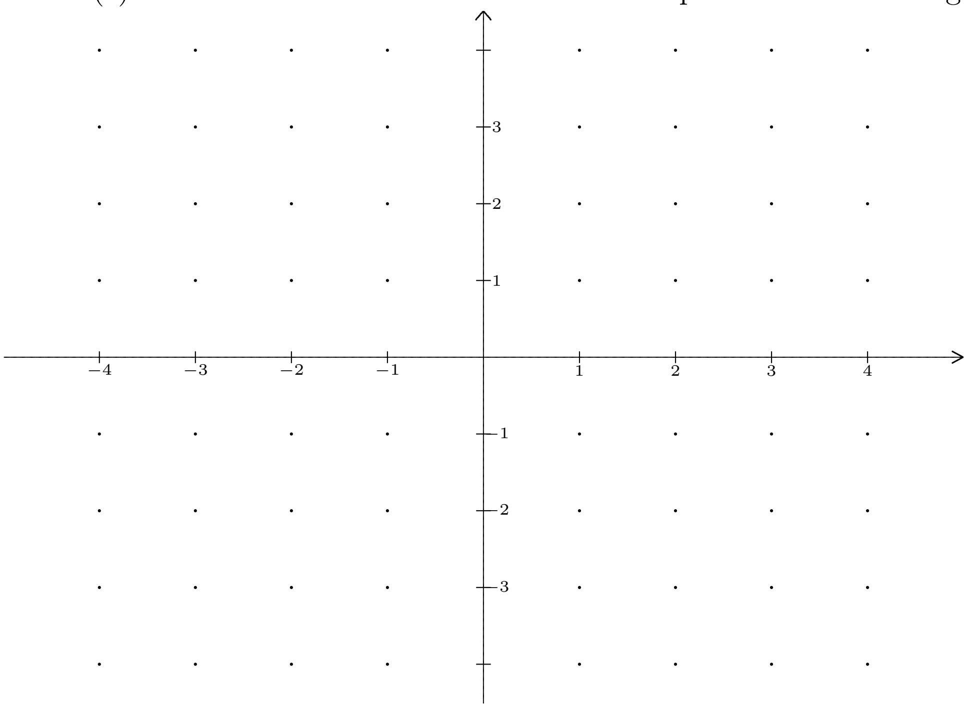
   (a) Build the sign chart for f

   Solution: We have to use all the points where f could change signs, $x = -3/2, 3, 0$, and 1. As expected the signs alternate starting with $+$ at the far left: $+ - + - +$.

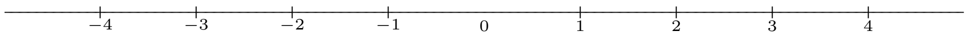
   (b) Find the vertical and horizontal asymptotes and the zeros, being careful not to mix them up.

   Solution: The zeros are $x = -3/2$ and $x = 3$ and the vertical asymptotes are $x = 0$ and $x = 1$. The horizontal asymptote is $y = 2$.

   (c) Use the information from the first two parts to sketch the graph of f.



   (d) From the graph, you can speculate on the existence of critical points if there are any. Write a sentence about where you expect to find these critical points or why you think there are none. Estimate the sign chart for $r'(x)$



Solution: Based on the graph, f has one critical point and it is in the interval $(0, 1)$. Suppose it is α. The f' is negative on $(-\infty, 0)$, negative on $(0, \alpha)$, positive on $(\alpha, 1)$, and positive on $(1, \infty)$.

3. (20 points) Consider the line L given by $y = 2x$, the point $P = (-4, 2)$, and the circle whose equation is $x^2 - 8x + y^2 - 4y = -16$.

   (a) Find the point on the line that is closest to the point P.

   Solution: The slope of the line L is 2 so the slope of the line joining P with the point on L closest to P is $-1/2$. The line through P with slope $-1/2$ goes through the origin, a point of L.

   (b) Find the point on the circle that is closest to the point P.

   Solution: The circle has center $(4, 2)$ and the radius is 2 which we find by completing the square. So the point of the circle closest to P is $(2, 2)$

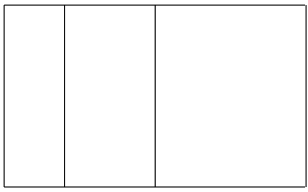
   (c) Find the point on the line that is closest to the circle.

   Solution: The line through the circles center with slope $-1/2$ is given by $y - 2 = -\frac{1}{2}(x - 4)$, and this line hits L at $(8/5, 16/5)$.

4. (10 points) Find a quadratic polynomial $f(x)$ which is decreasing on $(-\infty, 2)$ and increasing on $(2, \infty)$.

   Solution: The derivative f' must have zero $x = 2$, so $f'(x) = a(x-2)$. Letting $a = 1$, we have $f(x) = \frac{1}{2} \cdot x^2 - 2x + c$ for some constant c. Another solution is $f(x) = x^2 - 4x$. No calculus needed here.

5. (15 points) A farmer has 20000 feet of fencing to build a rectangular pasture. But he must separate the goats, horses and cows into different parts of the pasture using two vertical straight sections of fence as shown.



What is the area of the largest pasture the farmer can build?

Solution: Label the vertical pieces x and the horizontal pieces y. Then $2y + 4x = 20000$. To maximize the area, write $A = xy = x(5000 - x/2)$. Then $A'(x) = -x + 5000$ which has a zero at $x = 5000$. We can see that this maximizes A, and the maximum value os $A = 5000 \cdot 2500 = 12500000$.

6. (35 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	1	4

(a) $Q(x) = f(x)/g(x)$. Find $Q(5)$ and $Q'(5)$.
Solution: $Q'(x) = (f'(x)g(x) - g'(x)f(x))/(g(x))^2$. Therefore, $Q'(5) = (f'(5)g(5) - g'(5)f(5))/(g(5))^2 = \frac{3\cdot4-1\cdot5}{4^2} = \frac{7}{16}$.

(b) Let $H(x) = f(x)\cdot(g(x)+1)$. Compute $H(4)$ and $H'(4)$.
Solution: By the product and chain rules, $H'(x) = f'(x)\cdot(g(x)+1) + g'(x)\cdot f(x)$. Therefore, $H'(4) = f'(4)\cdot(g(4)+1)+g'(4)\cdot f(4) = 5\cdot3+6\cdot3 = 33$.

(c) Let $W(x) = f(g(x)+1)$. Compute both $W(5)$ and $W'(5)$.
Solution: Again by the chain rule, $W'(x) = f'(g(x)+1)\cdot g'(x)$, so $W'(5) = f'(g(5)+1)\cdot(g'(5)) = 3\cdot1 = 3$.

(d) Let $L(x) = g(\frac{1}{x}+1)$. Find $L(1)$ and $L'(1)$.
Solution: By the chain rule, $L'(x) = g'(\frac{1}{x}+1)\cdot(-x^{-2})$, so $L'(2) = g'(\frac{1}{1}+1)\cdot(-1^{-2}) = g'(2)\cdot(-1) = -4$.

(e) Let $U(x) = \sqrt{g(2x)}$. Compute $U(3)$ and $U'(3)$.
Solution: By the chain rule, $U'(x) = \frac{1}{2}g(2x)^{-1/2}\cdot g'(2x)\cdot 2$, so $U'(3) = \frac{1}{2}g(6)^{-1/2}\cdot g'(6)\cdot 2 = \frac{1}{2}\cdot 1\cdot 4\cdot 2 = 4$.

(f) Let $Z(x) = g(2x - f(x))$. Compute $Z(4)$ and $Z'(4)$.
Solution: Again by the chain rule and the product rule, $Z'(x) = g'(2x - f(x))\cdot(2 - f'(x))$ so $Z'(4) = g'(8 - f(4))\cdot(2 - f'(4)) = g'(5)(2-5) = 1(-3) = -3$.

7. (30 points) Let $g(x) = (x^2 - 4)^2(2x + 1)^2$.

   (a) Find $g'(x)$.

   Solution: Using the chain and product rules, we can differentiate $g(x)$ to get

   $$g'(x) = 2(x^2 - 4)(2x)(2x + 1)^2 + 2(2x + 1) \cdot 2(x^2 - 4)^2.$$

   (b) Find all the x-intercepts (the zeros) of $g'(x)$. That is, find the critical points of g.

   Solution: Factor out $4(x^2 - 4)(2x + 1)$ from both terms to get $g'(x) = 4(x^2 - 4)(2x + 1)[x(2x + 1) + x^2 - 4] = 4(x^2 - 4)(2x + 1)[3x^2 + x - 4] = 4(x - 2)(x + 2)(2x + 1)(3x + 4)(x - 1)$, so there are five distinct critical points, $-2, -4/3, -1/2, 1$ and 2.

   (c) Build the sigh chart for $g'(x)$.

   Solution: $g'(x)$ is positive on the intervals $(-2, -4/3), (-1/2, 1)$ and $(2, \infty)$.

   (d) Use the sign chart for $g'(x)$ to classify each critical point of g found in part (a) as the location of (i) a local minimum, (ii) a local maximum, or (iii) an imposter.

   Solution: Since g is increasing over the intervals where g' is positive, it follows that $g(x)$ has a minimum at each of the points $-2, -1/2$, and 2, and maxima at the other two points $-4/3$ and 1.