October 13, 2004 **Name** ______________________________

The total number of points available is 135. Throughout this test, **show your work.**

1. (15 points) Let $f(x) = \sqrt{x^3+1}$.

   (a) Compute $f'(x)$
   Solution: $f'(x) = \frac{1}{2}(x^3+1)^{-1/2} \cdot 3x^2 = \frac{3x^2}{2\sqrt{x^3+1}}$.

   (b) What is $f'(2)$?
   Solution: $f'(2) = \frac{3 \cdot 2^2}{2\sqrt{2^3+1}} = 12/6 = 2$

   (c) Use the information in b. to find an equation for the line tangent to the graph of f at the point $(2, f(2))$.
   Solution: Since $f(2) = 3$, using the point-slope form leads to $y - 3 = f'(2)(x-2) = 2(x-2)$, so $y = 2x - 1$.

2. (12 points) Consider the function f defined by:

$$f(x) = \begin{cases} 3x - x^3 & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 2x^{2/3} & \text{if } x > 1 \end{cases}$$

   (a) Is f continuous at $x = 1$?
   Solution: Yes, the limits from the left and right are both 2, and so is the value of f at 1.

   (b) What is the slope of the line tangent to the graph of f at the point $(8, 8)$?
   Solution: To find $f'(8)$ first note that when x is near 8, $f(x) = 2x^{2/3}$ so $f'(x) = 2\frac{2}{3}x^{-1/3}$. Thus, $f'(8) = 2\frac{2}{3}8^{-1/3} = 2\frac{2}{3} \cdot \frac{1}{2} = \frac{2}{3}$.

   (c) Find $f'(-3)$
   Solution: To find $f'(-3)$, we must differentiate the part of f defined for $x < 1$. In this area, $f'(x) = 3 - 3x^2$, so $f'(-3) = 3 - 3(-3)^2 = -24$.

3. (15 points) KAM Industries makes ovens. The daily cost in dollars of producing x ovens is given by

$$C(x) = -0.06x^2 + 120x + 5000,$$

for x in the range 0 to 2000.

(a) What is the actual cost of manufacturing the 101^{st} oven? ...the 201^{st} oven?

Solution: $C(101 - C(100) = -0.06(101)^2 + 120(101) + 5000 - (-0.06(100)^2 + 120(100) + 5000) = -0.06(201) - 120 = 107.94$. Similarly, $C(201) - C(200) = 95.94$.

(b) Find the marginal cost function $C'(x)$. What are $C'(100)$ and $C'(200)$?
Solution: $C'(x) = 120 - 0.12x$. $C'(100) = 108$ and $C'(200) = 96$.

(c) Find the average cost function $\overline{C}(x)$.
Solution: $\overline{C}(x) = \frac{-0.06x^2+120x+5000}{x} = \frac{5000}{x} + 120 - 0.06x = 5000x^{-1} + 120 - 0.06x$.

(d) Find the marginal average cost function $\overline{C}'(x)$.
Solution: $\overline{C}'(x) = -1(5000x^{-2}) - 0.06$.

4. (36 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $K(x) = f(x) \cdot g(x)$. Compute $K'(3)$

Solution: $K'(x) = f'(x) \cdot g(x) + g'(x)f(x)$, so $K'(3) = f'(3) \cdot g(3) + g'(3)f(3) = 2 \cdot 5 + 3 \cdot 1 = 13$.

(b) Let $L(x) = f(x)/g(x)$. Compute $L'(2)$.

Solution: $L'(x) = (f'(x)g(x) - g'(x)f(x)) \div (g(x))^2$, so $L'(2) = (f'(2)g(2) - g'(2)f(2)) \div (g(2))^2 = (4 \cdot 3 - 4 \cdot 6) \div 4^2 = -12/9 = -4/3$.

(c) Let $U(x) = f \circ g(x)$. Compute $U'(1)$.

Solution: By the chain rule, $U'(x) = f'(g(x)) \cdot g'(x)$, so $U'(1) = f'(g(1)) \cdot g'(1) = f'(2) \cdot g'(1) = 4 \cdot 5 = 20$.

(d) Let $V(x) = g \circ f(x)$. Compute $V'(5)$.

Solution: Again, by the chain rule, $V'(x) = g'(f(x)) \cdot f'(x)$, so $V'(5) = g'(f(5)) \cdot f'(5) = g'(5) \cdot f'(5) = 1 \cdot 3 = 3$.

(e) Let $W(x) = f(x^2 - g(x))$. Compute $W'(2)$.

Solution: Again by the chain rule, $W'(x) = f'(x^2 - g(x)) \cdot (2x - g'(x))$, so $W'(2) = f'(4 - g(2)) \cdot (4 - g'(2) = f'(4 - 3) \cdot (4 - 4) = 0$

(f) Let $Z(x) = g(x - f(x))$. Compute $Z'(3)$.

Solution: Again by the chain rule and the product rule, $Z'(x) = g'(x - f(x)) \cdot \frac{d}{dx}(x - f(x)) = g'(x - f(x)) \cdot (1 - f'(x))$, so $Z'(3) = g'(3 - f(3)) \cdot (1 - f'(3)) = g'(3 - 1) \cdot (1 - 2) = -4$.

5. (30 points) Compute the following derivatives.

   (a) Let $f(x) = x^{-2} + \sqrt{x}$. Find $\frac{d}{dx}f(x)$.
   Solution: $f'(x) = -2x^{-3} + \frac{1}{2}x^{-\frac{1}{2}}$.

   (b) Let $g(x) = \sqrt{x^4 - x^2}$. What is $g'(x)$?
   Solution: $g'(x) = \frac{1}{2}(x^4 - x^2)^{-\frac{1}{2}} \cdot (4x^3 - 2x) = (4x^3 - 2x) \div 2\sqrt{x^4 - x^2} = (2x^3 - x) \div \sqrt{x^4 - x^2}$.

   (c) Find $\frac{d}{dx}((4x+1)^2 \cdot (2x^3 - 1))$.
   Solution: By the product rule, $\frac{d}{dx}((4x+1)^2 \cdot (2x^3-1)) = 2(4x+1) \cdot 4(2x^3-1) + 6x^2(4x+1)^2 = 2(4x+1)[(8x^3-4) + 3x^2(4x+1)] = 2(4x+1)[20x^3 + 3x^2 - 4]$.

   (d) Find $\frac{d}{dx}\frac{2x^2+1}{x-2}$.
   Solution: By the quotient rule, $\frac{d}{dx}\frac{2x^2+1}{x-2} = \frac{4x(x-2)-1(2x^2+1)}{(x-2)^2} = \frac{2x^2-8x-1}{x^2-4x+4}$.

   (e) Find $\frac{d}{dt}(t^2 + 1/t)^4$.
   Solution: By the chain rule, $\frac{d}{dt}(t^2 + 1/t)^4 = 4(t^2 + 1/t)^3 \cdot (2t - t^{-2})$.

6. (10 points) Let $f(x) = \sqrt{2x-3}$. The chain rule can be applied to find that $f'(x) = \frac{1}{2}(2x-3)^{-1/2} \cdot 2 = 1/\sqrt{2x-3}$. Use the limit definition of derivative to verify this fact.

   Solution: We need to compute $\lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$. Thus,

$$
\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{\sqrt{2(x+h)-3} - \sqrt{2x-3}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{2(x+h)-3} - \sqrt{2x-3}}{h} \cdot \frac{\sqrt{2(x+h)-3} + \sqrt{2x-3}}{\sqrt{2(x+h)-3} + \sqrt{2x-3}} \\
&= \lim_{h \to 0} \frac{2(x+h)-3-(2x-3)}{h\sqrt{2(x+h)-3} + \sqrt{2x-3}} \\
&= \lim_{h \to 0} \frac{2x+2h-3-2x+3}{h\sqrt{2(x+h)-3} + \sqrt{2x-3}} \\
&= \lim_{h \to 0} \frac{2h}{h\sqrt{2(x+h)-3} + \sqrt{2x-3}} \\
&= \frac{2}{\sqrt{2x-3} + \sqrt{2x-3}} = 1/\sqrt{2x-3}
\end{aligned}
$$

7. (12 points) Intermediate Value Theorem. Recall that the IVT asserts the following: If f is a continuous function on the interval $[a, b]$ and M is a number between $f(a)$ and $f(b)$, then there exists a number c satisfying $a \le c \le b$ and $f(c) = M$. For this problem let $f(x) = \sqrt{2x-5}$ and let $[a, b] = [3, 15]$. Finally, suppose $M = 4$. Find the number c whose existence is guaranteed by IVT.

   Solution: We need to solve the equation $\sqrt{2x-5} = 4$ for x. Square both sides to get $2x - 5 = 16$, from which it follows that $x = 21/2$.