**July 21, 2005** **Name** ______________________________

The total number of points available is 122. Throughout this test, **show your work.**

1. (10 points) Consider the parabola $f(x) = 3x^2 + 2x + 2$.

   (a) What is the slope of the line tangent to the graph of $f$ at the point $(0, 2)$?
   **Solution:** $f'(x) = 6x + 2$ so $f'(0) = 2$. The slope of the tangent line is 2.

   (b) Write an equation of this tangent line in the form $y = mx + b$.
   **Solution:** Since the slope is 2, we can use the point slope form to get $y - 2 = 2(x - 0) = 2x$, so $y = 2x + 2$.

2. (12 points) The point $P(3, 19)$ lies on the curve $y = x^2 + x + 7$. If $Q$ is the point $(x, x^2 + x + 7)$, find the slope of the secant line $PQ$ for the following values of $x$.

   (a) If $x = 3.1$, the slope of $PQ$ is:
   **Solution:** Using a calculator, $(3.1^2 + 3.1 + 7 - 19) \div (3.1 - 3) = 7.1$

   (b) If $x = 3.01$, the slope of $PQ$ is:
   **Solution:** Using a calculator, $(3.01^2 + 3.01 + 7 - 19) \div (3.01 - 3) = 7.01$

   (c) If $x = 2.9$, the slope of $PQ$ is:
   **Solution:** Using a calculator, $(2.9^2 + 2.9 + 7 - 19) \div (2.9 - 3) = 6.9$

   (d) If $x = 2.99$, the slope of $PQ$ is:
   **Solution:** Using a calculator, $(2.99^2 + 2.99 + 7 - 19) \div (2.99 - 3) = 6.99$

   (e) Based on the above results, guess the slope of the tangent line to the curve at $P(3, 19)$.
   **Solution:** The obvious guess is $y' = 7$ at $x = 3$ and that is true since $y' = 2x + 1$ for any $x$.

3. (10 points) Intermediate Value Theorem. Recall that the IVT asserts the following: If $f$ is a continuous function on the interval $[a, b]$ and $M$ is a number between $f(a)$ and $f(b)$, then there exists a number $c$ satisfying $a \leq c \leq b$ and $f(c) = M$. For this problem let $f(x) = \sqrt{2x - 2}$ and let $[a, b] = [1, 3]$. Finally, suppose $M = 1$. Find the number $c$ whose existence is guaranteed by the IVT.

**Solution:** We need to solve the equation $\sqrt{2x-2} = 1$ for $x$. Square both sides to get $2x - 2 = 1$, from which it follows that $x = 3/2$.

4. (15 points) Let $f(x) = 2/x$.

   (a) Construct $\frac{f(3+h)-f(3)}{h}$

   **Solution:** $f(3+h) = 2/(3+h)$ and of course $f(3) = 2/3$, so $\frac{f(3+h)-f(3)}{h} = (2/(3+h) - 2/3) \div h$.

   (b) Simplify and take the limit of the expression in (a) as $h$ approaches 0 to find $f'(3)$.

   **Solution:** Take the limit of $\frac{f(3+h)-f(3)}{h} = (2/(3+h) - 2/3) \div h$ as $h \to 0$ to get $f'(3) = -2/9$. You can check this answer using the power rule: $f(x) = 2/x = 2x^{-1}$, so $f'(x) = 2(-1)x^{-2}$.

   (c) Use the information found in (b) to find an equation for the line tangent to the graph of $f$ at the point $(3, 2/3)$.

   **Solution:** Use the point-slope form to get $y - 2/3 = -2/9(x-3)$ which reduces to $y = -2x/9 + 4/3$.

5. (30 points) Recall that $\sqrt{x}$ is a well-defined real number if and only if $x \geq 0$. Use this fact to find the domain of the function $g(x)$ defined by

$$g(x) = \sqrt{(x-5)(x-3)(x)(x+1)^2(x+4)}.$$

It's important to show all your work, including the test points and the matrix of values of the factors at the test points.

**Solution:** Let $u(x) = (x-5)(x-3)(x)(x+1)^2(x+4)$. We need to solve the inequality $u(x) \geq 0$. To this end, note that the roots of $u(x) = 0$ are $x = 5, x = 3, x = 0, x = -1$, and $x = -4$. These breakpoints split the line into six intervals, $(\infty, -4), (-4, -1), (-1, 0), (0, 3), (3, 5)$, and $(5, \infty)$. Using test points $-5, -2, -1/2, 1, 4$, and 6, we find that $u(-5) > 0, u(-2) < 0, u(-1/2) < 0, u(1) > 0, u(4) < 0, u(6) > 0$. Notice that although $x = -1$ is the endpoint of two abutting intervals over which $g$ is negative, $g(-1) = 0$. Hence we can write the domain of $g$ as $(\infty, -4] \cup [0, 3] \cup [5, \infty) \cup \{-1\}$.

6. (15 points) Let $F(x) = f(x^3)$ and $G(x) = (f(x))^3$. You also know that $a^2 = 10, f(a) = 3, f'(a) = 14, f'(a^3) = 2$.

   (a) Find $F'(a)$.
   **Solution:** By the chain rule with $f$ as the outside function and $x^3$ as the inside function, $F'(x) = f'(x^3) \cdot 3x^2$. Therefore $F'(a) = 2 \cdot 3 \cdot 10 = 60$.

   (b) Find $G'(a)$.
   **Solution:** By the chain rule with $x^3$ as the outside function and $f$ as the inside function, $G'(x) = 3(f(x))^2 \cdot f'(x)$. Therefore $G'(a) = 3f(a)f'(a) = 3 \cdot 9 \cdot 14 = 378$.

7. (30 points) Compute the following derivatives.

   (a) Let $f(x) = x^2 + x^{-\frac{2}{3}}$. Find $\frac{d}{dx}f(x)$.
   **Solution:** $\frac{d}{dx}f(x) = 2x - 2x^{-\frac{5}{3}}/3 = 2x - \frac{2}{3x^{\frac{5}{3}}}$.

   (b) Let $g(x) = \sqrt{x^3 + x + 4}$. What is $g'(x)$?
   **Solution:** $g'(x) = 1/2(x^3 + x + 4)^{-1/2} \cdot (3x^2 + 1) = \frac{3x^2+1}{2\sqrt{x^3+x+4}}$.

   (c) Find $\frac{d}{dx}((3x+1)^2 \cdot (4x^2 - 1))$

   (d) Let $f(x) = (2x^2 + 1)^4$. Find $f''(x)$.
   **Solution:** Note that, by the chain rule, $f'(x) = 4(2x^2 + 1)^3 \cdot 4x = 16x(2x^2 + 1)^3$. To find $f''(x)$, we must use the chain rule and the product rule. Thus $f''(x) = 16(2x^2 + 1)^3 + 3(2x^2 + 1)^2 \cdot 4x \cdot 16x = 16(2x^2 + 1)^3 + 192x^2(2x^2 + 1)^2 = 144x^3 + 72x^2 - 10x - 6$.

(e) Find $\frac{d}{dt}(t^3 + 1/t)^2$.
**Solution:** $\frac{d}{dt}(t^3 + 1/t)^2 = 2(t^3 + t^{-1})(3t^2 - t^{-2})$.