March 1, 2007 **Name** ______________________________

The total number of points available is 131. Throughout this test, **show your work.**

1. (12 points) Let $f(x) = \sqrt{x^4 - 2x + 5}$.

   (a) Compute $f'(x)$

   Solution: $f'(x) = \frac{1}{2}(x^4 - 2x + 5)^{-1/2}(4x^3 - 2) = \frac{4x^3-2}{2\sqrt{x^4-2x+5}}$.

   (b) What is $f'(1)$?

   Solution: $f'(1) = \frac{4-2}{2\sqrt{1-2+5}} = 1/2$

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(1, f(1))$.

   Solution: Since $f(1) = 2$, using the point-slope form leads to $y - 2 = f'(2)(x - 1) = (x - 1)/2$, so $y = x/2 + 3/2$.

2. (12 points) For what values of x is the tangent line of the graph of

$$f(x) = 2x^3 - 3x^2 - 72x + 12$$

   parallel to the line $y = -60x + 7$?

   Solution: Since $f'(x) = 6x^2 - 6x + 72 = 6(x^2 - x - 12)$, we seek those values of x such that $6(x^2 - x - 12) = -60$. Factoring and simplifying yields $x^2 - x - 12 = -10$ and $(x - 2)(x + 1) = 0$, so the two values of are $x = -1$ and $x = 2$.

3. (32 points) The cost of producing widgets is given by $C(x) = 10000 + 40x - 0.001x^2, \quad 0 \leq x \leq 1000$. The relationship between price and demand for widgets is given by $p = f(x) = -0.02x + 300, \quad 0 \leq x \leq 7000$.

   (a) Find the average cost function $\overline{C}(x)$.
   Solution: $\overline{C}(x) = 10000/x + 40 - 0.001x$.

   (b) Find the (incremental) cost of producing the 500^{th} widget.
   Solution: $C(500) - C(499) = 10000 + 40 \cdot 500 - 0.001 \cdot 500^2 - (10000 + 40 \cdot 499 - 0.001 \cdot 499^2) = 40 - 0.001(999) = 39.001$.

   (c) Find the marginal cost function $C'(x)$.
   Solution: $C'(x) = 40 - 0.002x$.

   (d) What is $C'(500)$?
   Solution: $C'(500) = 40 - 0.002(500) = 40 - 1 = 39$.

   (e) Find the marginal average cost function $\overline{C}'(x)$.
   Solution: $\overline{C}'(x) = -10000/x^2 - 0.001$.

   (f) Find the revenue function $R(x)$.
   Solution: $R(x) = xp = xf(x) = x(-0.2x + 300) = -.02x^2 + 300x$.

   (g) Find the marginal revenue function $R'(x)$.
   Solution: $R'(x) = -0.04x + 300$.

   (h) Find the profit function $P(x)$.
   Solution: $P(x) = R(x) - C(x) = 260x - 0.019x^2 - 10000$.

   (i) Find the marginal profit function $P'(x)$.
   Solution: $P'(x) = 260 - 0.038x$.

   (j) Find a value of x where the profit function $P(x)$ has a horizontal tangent line.
   Solution: Solve $P'(x) = 0$ for x to get $x = 260 \div 0.038 \approx 6,842$.

4. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = \frac{f(x)}{g(x)}$. Compute $L'(2)$.

Solution: $L'(x) = (f'(x)g(x) - g'(x)f(x)) \div g(x)^2$, so $L'(2) = (f'(2)g(2) - g'(2)f(2)) \div g(2)^2 = (4 \cdot 3 - 4 \cdot 6) \div 9 = -4/3$.

(b) Let $U(x) = g \circ f(x)$. Compute $U'(1)$.

Solution: By the chain rule, $U'(x) = g'(f(x)) \cdot f'(x)$, so $U'(1) = g'(f(1)) \cdot f'(1) = g'(4) \cdot f'(1) = 6 \cdot 6 = 36$.

(c) Let $K(x) = g(x^2) + f(x)$. Compute $K'(1)$

Solution: $K'(x) = g'(x^2) \cdot 2x + f'(1)$, so $K'(1) = g'(1) \cdot 2 \cdot 1 + f'(1) = 5 \cdot 2 + 6 = 16$.

(d) Let $V(x) = f(g(x^2))$. Compute $V'(2)$.

Solution: Again, by the chain rule, $V'(x) = f'(g(x^2)) \cdot g'(x^2) \cdot 2x$, so $V'(2) = f'(g(4)) \cdot g'(4) \cdot 4 = f'(2) \cdot g'(4) \cdot 2 = 4 \cdot 6 \cdot 4 = 96$.

(e) Let $W(x) = g(f(x) - x)$. Compute $W'(5)$.

Solution: Again by the chain rule, $W'(x) = g'(f(x) - x) \cdot (f'(x) - 1)$, so $W'(5) = g'(f(5) - 5) \cdot (f'(5) - 1) = g'(0)(3 - 1) = 2 \cdot 2 = 4$.

(f) Let $Z(x) = f(3x - f(x))$. Compute $Z'(1)$.

Solution: Again by the chain rule and the product rule, $Z'(x) = f'(3x - f(x)) \cdot (3 - f'(x)$ so $Z'(1) = f'(3 - f(1)) \cdot (3 - f'(1)) = f'(-1)(3 - 6) = -3f'(-1)$.

5. (20 points) Compute the following derivatives.

   (a) Let $f(x) = (1 + \sqrt{1 + x^4})^2$. Find $\frac{d}{dx} f(x)$.

   Solution: Note that $f'(x) = 2(1 + \sqrt{1 + x^4}) \cdot 1/2(1 + x^4)^{-1/2} \cdot (4x^3) = 4x^3(1 + sqrt1 + x^4) \div \sqrt{1 + x^4}$.

   (b) Let $g(x) = x^2/(1 + x^3)$. What is $g'(x)$?

   Solution: Use the quotient rule to get $g'(x) = 2x(1 + x^3) - 3x^2(x^2) \div (1 + x^3)^2 = \dfrac{-x^4 + 2x}{(1 + x^3)^2}$.

   (c) Find $\frac{d}{dx}((x - 2)^2 \cdot (3x - 1))$.

   Solution: By the product rule, $\frac{d}{dx}((x - 2)^2 \cdot (3x - 1)) = 2(x - 2) \cdot (3x - 1) + 3(x - 2)^2 = (x - 2)(9x - 8)$.

   (d) Find $\frac{d}{dx} \sqrt{\frac{2x^2+1}{3x+2}}$.

   Solution: By the chain and quotient rules, $\frac{d}{dx} \dfrac{2x^3 + 1}{x - 2} = \dfrac{1}{2} \left(\dfrac{2x^2 + 1}{3x + 2} \right)^{-1/2} \cdot \dfrac{4x(3x + 2) - 3(2x^2 + 1)}{(3x + 2)^2}$.

   (e) Find $\frac{d}{dt}(t^{-2} - t^{2/3})$.

   Solution: By the chain rule, $\frac{d}{dt} \frac{d}{dt}(t^{-2} - t^{2/3}) = -2t^{-3} - \frac{2}{3} t^{-1/3}$.

6. (25 points) Find the domain of the function

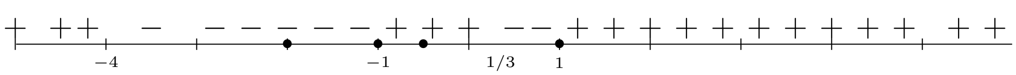
$$f(x) = \sqrt{\frac{(x-4)(x+4)(3x-1)}{(3x^2-3)(x-4)}}.$$

Express your answer in interval form.

Solution: Notice first that f is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$r(x) = \frac{(x-4)(x+4)(3x-1)}{(3x^2-3)(x-4)}.$$

We can cancel the common factors with the understanding that we are (very slightly) enlarging the domain of r: $r(x) = \frac{(x+4)(3x-1)}{3(x-1)(x+1)}$. Next find the branch points. These are the points at which f can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $-4, 1/3, 1, -1$. Again we select test points and find the sign of f at of these points to get the sign chart.



Again suppose that we are solving $f(x) \geq 0$ The solution to $f(x) > 0$ is easy. It is the union of the open intervals with the + signs, $(-\infty, -4) \cup (-1, 1/3) \cup (1, \infty)$. It remains to solve $f(x) = 0$ and attach these solutions to what we have. The zeros of f are -4 and $1/3$. So the solution to $f(x) \geq 0$ is $(-\infty, -4] \cup (-1, 1/3] \cup (1, \infty)$. Notice that the branch points 1 and -1 are not included since f is not defined at these two points. It has vertical asymptotes at these two places. Technically the value $x = 4$ should not be included in the solution because the function f as originally defined is not defined at $x = 4$. Thus, the exact answer is $(-\infty, -2] \cup (-1, 1/3] \cup (1, 4) \cup (4, \infty)$.