October 25, 2007 **Name** ______________________________

The total number of points available is 131. Throughout this test, **show your work.**

1. (12 points) Let $f(x) = \sqrt{x^2 - x + 3}$.

   (a) Compute $f'(x)$

   Solution: $f'(x) = \frac{1}{2}(x^2 - x + 3)^{-1/2} \cdot 2x - 1 = \frac{2x-1}{2\sqrt{x^2-x+3}}$.

   (b) What is $f'(3)$?

   Solution: $f'(3) = \frac{2 \cdot 3 - 1}{2 \cdot 9^{1/2}} = 5/6$

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(3, f(3))$.

   Solution: Since $f(3) = 3$, using the point-slope form leads to $y - 3 = f'(3)(x - 3) = 5(x - 3)/6$, so $y = 5x/6 + 1/2$.

2. (12 points) Consider the function f defined by:

$$f(x) = \begin{cases} \sqrt{x+3} & \text{if } x < 1 \\ 2 & \text{if } x = 1 \\ 2(x-2)^2 & \text{if } x > 1 \end{cases}$$

   (a) Is f continuous at $x = 1$? Your answer must make clear that you know and understand the definition of continuity. A yes/no correct answer is worth 1 point.

   Solution: Yes, the limits from the left and right are both 2, and the value of f at 1 is 2, so $\lim_{x \to 1} f(x) = f(1)$.

   (b) What is the slope of the line tangent to the graph of f at the point $(8, 72)$?

   Solution: To find $f'(8)$ first note that when x is near 8, $f(x) = 2(x-2)^2$ so $f'(x) = 4(x - 2)$. Thus, $f'(8) = 4(8 - 2)24$.

   (c) Find $f'(-2)$

   Solution: To find $f'(-2)$, we must differentiate the part of f defined for $x < 1$. In this area, $f'(x) = (x + 3)^{-1/2}/2$, so $f'(-2) = 1/2$.

3. (18 points) If a ball is thrown vertically upward from the roof of 128 foot building with a velocity of 64 ft/sec, its height after t seconds is $s(t) = 128 + 64t - 16t^2$.

   (a) What is the height the ball at time $t = 1$?

   Solution: $s(1) = 176$.

   (b) What is the velocity of the ball at the time it reaches its maximum height?

   Solution: $s'(t) = v(t) = 0$ when the ball reaches its max height.

   (c) What is the maximum height the ball reaches?

   Solution: Solve $s'(t) = 64 - 32t = 0$ to get $t = 2$ when the ball reaches its zenith. Thus, the max height is $s(2) = 128 + 64(2) - 16(2)^2 = 192$.

   (d) After how many seconds is the ball exactly 160 feet above the ground?

   Solution: Use the quadratic formula to solve $128 + 64t - 16t^2 = 160$. You get $t = \frac{4 \pm \sqrt{16-8}}{2} = 2 \pm \sqrt{2}$.

   (e) How fast is the ball going the first time it reaches the height 160?

   Solution: Evaluate $s(t)$ when $t = 2 - \sqrt{2}$ to get $32\sqrt{2}$.

   (f) How fast is the ball going the second time it reaches the height 160? olEvaluate $s(t)$ when $t = 2 + \sqrt{2}$to get $-32\sqrt{2}$. In other words the ball is going downward at the same rate it was moving upwards when first went through 160 feet.

4. (12 points) The cost of producing x units of stuffed alligator toys is $C(x) = 0.004x^2 + 8x + 6000$.

   (a) Find the marginal cost at the production level of 1000 units.

   Solution: $C'(x) = \frac{d}{dx}0.004x^2 + 8x + 6000 = 0.008x + 8$ so $C'(1000) = 16$.

   (b) What is the marginal average cost function?

   Solution: $\overline{C}(x) = 0.004x + 8 + 6000x^{-1}$, so $\overline{C}'(x) = 0.004 - 6000x^{-2}$.

   (c) What is $\overline{C}'(500)$? Interpret your answer.

   Solution: $\overline{C}(500) = -0.02$, which means the average cost is decreasing when the production level is 500.

5. (30 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	6	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = f(x)/g(x)$. Compute $L'(2)$.

Solution: Use the quotient rule. $L'(x) = \frac{f'(x)g(x)-g'(x)f(x)}{g(x)^2}$, so $L'(2) = (f'(2)g(2) - g'(2)f(2)/g(2)^2 = (4 \cdot 3 - 4 \cdot 6)/9 = -4/3$.

(b) Let $U(x) = f \circ f(x)$. Compute $U'(1)$.

Solution: By the chain rule, $U'(x) = f'(f(x)) \cdot f'(x)$, so $U'(1) = f'(f(1)) \cdot f'(1) = f'(4) \cdot f'(1) = 5 \cdot 6 = 30$.

(c) Let $K(x) = \sqrt{f(x)}$. Compute $K'(1)$

Solution: $K'(x) = f(x)^{-1/2}/2 \cdot f'(x)$. So $K'(1) = 1/2sqrtf(1) \cdot f'(1) = 6/4 = 3/2$.

(d) Let $V(x) = x^2(g(2x))$. Compute $V'(3)$.

Solution: Again, by the product rule and the chain rule, $V'(x) = 2x(g(2x)) + x^2g'(2x) \cdot 2$, so $V'(3) = 2 \cdot 3g(6) + 18g'(6) = 6 \cdot 2 + 18 \cdot 4 = 12 + 72 = 84$.

(e) Let $W(x) = [g(x - f(x))]^3$. Compute $W'(4)$.

Solution: Again by the chain rule, $W'(x) = 3g(x - f(x))^2 \cdot g'(x - f(x)) \cdot (1 - f'(x))$, so $W'(4) = 3g(4 - f(4))^2 \cdot g'(4 - f(4)) \cdot (1 - f'(4)) = 3g(1) \cdot g'(1) \cdot (1 - 5) = 3 \cdot 4 \cdot 5(1 - 5) = -240$.

(f) Let $Z(x) = f(x^2 + g(x))$. Compute $Z'(1)$.

Solution: Again by the chain rule and the product rule, $Z'(x) = f'(x^2 + g(x)) \cdot \frac{d}{dx}(x^2 + g(x)) = f'(x^2 + g(x)) \cdot (2x + g'(x))$, so $Z'(1) = f'(1 + g(1)) \cdot (2 \cdot 1 + g'(1)) = f'(3) \cdot (2 + 5) = 2 \cdot 7 = 14$.

6. (15 points) Compute the following derivatives.

   (a) Let $f(x) = (x + \sqrt{1 + x^3})^4$. Find $\frac{d}{dx} f(x)$.
   Solution: We differentiate it using the power rule and chain rule: $f'(x) = 4(x + \sqrt{1 + x^3})^3 \cdot (1 + (\frac{1}{2}(1 + x^3)^{-1/2} \cdot 3x^2))$.

   (b) Let $g(x) = x^2/(1 + x^2)$. What is $g'(x)$?
   Solution: Use the quotient rule to get $g'(x) = 2x(1+x^2) - 2x(x^2) \div (1+x^2)^2 = \dfrac{2x}{(1 + x^2)^2}$.

   (c) Find $\frac{d}{dx} \sqrt{\frac{2x^3+1}{3x-2}}$.

   Solution: By the chain and quotient rules, $\frac{d}{dx} \dfrac{2x^3 + 1}{3x - 2} = \dfrac{1}{2} \left(\dfrac{2x^3 + 1}{3x - 2} \right)^{-1/2} \cdot \dfrac{6x^2(3x - 2) - 3(2x^3 + 1)}{(3x - 2)^2}$. This expressions simplifies to one with a numerator $12x^3 - 12x^2 - 3$.

7. (12 points) Find all critical points of $f(x) = ((x+2)^2 \cdot (2x-1))$. Then identify each critical point as the location of a local maximum, local minimum, or neither.

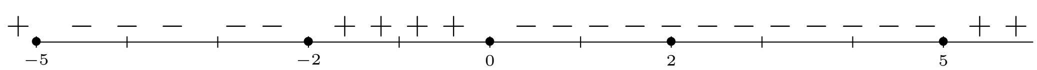
   Solution: By the product rule, $f'(x) = (2(x + 2) \cdot (2x - 1)) + 2(x + 2)^2 = 2(x+2)[(2x-1)+(x+2)] = 2(x+2)(3x+1)$, so the stationary points are $x = -2$ and $x = -1/3$. Since $f'(x)$ is negative between these two critical points, it follows that f has a local max at $x = -2$ and a local min at $x = -1/3$.

8. (20 points) Suppose a function f has been differentiated to give

$$f'(x) = (x^2 - 4)(x)(3x^2 - 75)(x - 2).$$

Use the Test Interval Technique on f' to find the sign chart for f'. Then list in interval notation the intervals over which the function f is increasing.

Solution: Notice first that f' is not in factored form. Factoring reveals $f'(x) = (x^2 - 4)(x)(3x^2 - 75)(x - 2) = 3(x - 2)^2(x)(x + 2)(x - 5)(x + 5)$. Next find the branch points. These are the points at which f' is zero. They are $-2, 2, 0, 5, -5$. Again we select test points and find the sign of f at of these points to get the sign chart.



So you can see from the sign chart that f is increasing on all three of the intervals $(-\infty, -5], [-2, 0]$ and $[5, \infty)$. Of course, using (and) instead of [and] is fine.