February 28, 2008 **Name** ________________________________

The total number of points available is 137. Throughout this test, **show your work.**

1. (15 points) Let $f(x) = \sqrt{x^4 - 3x + 11}$.

   (a) Compute $f'(x)$
   Solution: $f'(x) = \frac{1}{2}(x^4 - 3x + 11)^{-1/2}(4x^3 - 3) = \frac{4x^3-3}{2\sqrt{x^4-3x+11}}$.

   (b) What is $f'(1)$?
   Solution: $f'(1) = \frac{4-3}{2\sqrt{1-3+11}} = 1/6$

   (c) Use the information in (b) to find an equation for the line tangent to the graph of f at the point $(1, f(1))$.
   Solution: Since $f(1) = 2$, using the point-slope form leads to $y - 3 = f'(1)(x - 1) = (x - 1)/6$, so $y = x/6 + 17/6$.

2. (15 points) For what values of x is the line tangent to the graph of

$$f(x) = (2x + 1)^2(3x - 4)^2$$

   parallel to the line $y = 7$?

   Solution: Since $f'(x) = 2(2x + 1) \cdot 2 \cdot (3x - 4)^2 + 2(3x - 4) \cdot 3 \cdot (2x + 1)^2$, we seek those values of x such that $2(2x + 1)(3x - 4)[2(3x - 4) + 3(2x + 1)] = 0$. Factoring and simplifying yields $x = -1/2, x = 4/3$ and $6x - 8 + 6x + 3 = 0$, so the three values of are $x = -1/2, x = 4/3$ and $x = 5/12$.

3. (32 points) The cost of producing widgets is given by $C(x) = 10000 + 50x - 0.003x^2, \quad 0 \le x \le 1000$. The relationship between price and demand for widgets is given by $p = f(x) = -0.04x + 300, \quad 0 \le x \le 7000$, where p is the price in dollars.

   (a) Find the average cost function $\overline{C}(x)$.
   Solution: $\overline{C}(x) = 10000/x + 50 - 0.003x$.

   (b) Find the (incremental) cost of producing the 500^{th} widget.
   Solution: $C(500) - C(499) = 10000 + 50 \cdot 500 - 0.001 \cdot 500^2 - (10000 + 50 \cdot 499 - 0.001 \cdot 499^2) = 50 - 0.003(999) = 47.003$.

   (c) Find the marginal cost function $C'(x)$.
   Solution: $C'(x) = 50 - 0.006x$.

   (d) What is $C'(500)$?
   Solution: $C'(500) = 50 - 0.006(500) = 47$.

   (e) Find the marginal average cost function $\overline{C}'(x)$.
   Solution: $\overline{C}'(x) = -10000/x^2 - 0.003$.

   (f) Find the revenue function $R(x)$.
   Solution: $R(x) = xp = xf(x) = x(-0.2x + 300) = -.04x^2 + 300x$.

   (g) Find the marginal revenue function $R'(x)$.
   Solution: $R'(x) = -0.08x + 300$.

   (h) Find the profit function $P(x)$.
   Solution: $P(x) = R(x) - C(x) = 250x - 0.037x^2 - 10000$.

   (i) Find the marginal profit function $P'(x)$.
   Solution: $P'(x) = 250 - 0.074x$.

   (j) Find a value of x where the profit function $P(x)$ has a horizontal tangent line.
   Solution: Solve $P'(x) = 0$ for x to get $x = 250 \div 0.074 \approx 3378$.

4. (20 points) Compute the following derivatives.

   (a) Let $f(x) = (1 + \sqrt{1 + x^2})^2$. Find $\frac{d}{dx} f(x)$.
   Solution: Note that $f'(x) = 2(1 + \sqrt{1 + x^2}) \cdot 1/2(1 + x^2)^{-1/2} \cdot (2x) = 2x(1 + \sqrt{1 + x^2}) \div \sqrt{1 + x^2}$.

   (b) Find $\frac{d}{dt}(t^{-3} - \sqrt{t^3})$.
   Solution: By the chain rule, $\frac{d}{dt}\frac{d}{dt}(t^{-2} - t^{2/3}) = -3t^{-4} - \frac{3}{2}t^{1/2}$.

   (c) Let $g(x) = x^2/(1 + x^2)$. What is $g'(x)$?
   Solution: Use the quotient rule to get $g'(x) = 2x(1+x^2) - 2x(x^2) \div (1 + x^2)^2 = \dfrac{2x}{(1 + x^2)^2}$.

   (d) Find $\frac{d}{dx}\sqrt{\frac{2x^2+1}{3x+2}}$.
   Solution: By the chain and quotient rules, $\frac{d}{dx}\dfrac{2x^3 + 1}{x - 2} = \left(\dfrac{2x^2 + 1}{3x + 2}\right)^{-1/2} \cdot \dfrac{6x^2 + 8x - 3}{2(3x + 2)^2}$.

5. (20 points) Find the domain of the function

$$f(x) = \sqrt{\frac{(x-2)(x+2)(3x-1)}{(3x^2-27)(x-2)}}.$$

Express your answer in interval form.

Solution: Notice first that f is not in factored form. Factoring reveals that the numerator and denominator have common factors. Thus

$$r(x) = \frac{(x-2)(x+2)(3x-1)}{3(x^2-9)(x-2)}.$$

We can cancel the common factors with the understanding that we are (very slightly) enlarging the domain of r: $r(x) = \frac{(x+2)(3x-1)}{3(x-3)(x+3}$. Next find the branch points. These are the points at which f can change signs. Precisely, they are the zeros of the numerator and of the denominator. They are $-3, -2, 1/3, 3$. Again we select test points and find the sign of f at of these points to get the sign chart. Again suppose that we are solving $f(x) \geq 0$ The solution to $f(x) > 0$ is easy. It is the union of the open intervals with the + signs, $(-\infty, -3) \cup (-2, 1/3) \cup (3, \infty)$. It remains to solve $f(x) = 0$ and attach these solutions to what we have. The zeros of f are -4 and $1/3$. So the solution to $f(x) \geq 0$ is $(-\infty, -3] \cup [-2, 1/3] \cup (3, \infty)$.

6. (35 points) Consider the table of values given for the functions f, f', g, and g':

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	2	1	3	2
1	4	6	2	5
2	6	4	3	4
3	1	2	5	3
4	3	5	2	6
5	5	3	4	1
6	0	3	2	4

(a) Let $L(x) = \frac{x+f(x)}{g(x)}$. Compute $L'(2)$.

Solution: $L'(x) = (1 + f'(x)g(x) - g'(x)(x + f(x)) \div g(x)^2$, so $L'(2) = (1+f'(2)g(2)-g'(2)(2+f(2))\div g(2)^2 = (1+4)\cdot 3-4\cdot(2+6)\div 9 = -17/9$.

(b) Let $U(x) = f \circ f(2x)$. Compute $U'(1)$.

Solution: By the chain rule, $U'(x) = 2f'(f(x)) \cdot f'(2x)$, so $U'(1) = 2f'(f(2)) \cdot f'(2) = 2f'(6) \cdot f'(2) = 2 \cdot 3 \cdot 4 = 24$.

(c) Let $K(x) = g(x^3) + f(x)$. Compute $K'(1)$

Solution: $K'(x) = g'(x^3) \cdot 3x^2 + f'(x)$, so $K'(1) = g'(1) \cdot 3 + f'(1) = 5 \cdot 3 + 6 = 21$.

(d) Let $Z(x) = g(2x - f(x))$. Compute $Z'(3)$. Be careful here with the parens. Note that the inside function is $2x - f(x)$.

Solution: Again by the chain rule and the product rule, $Z'(x) = g'(2x - f(x)) \cdot (2 - f'(x))$ so $Z'(3) = g'(6 - f(3)) \cdot (2 - f'(3)) = g'(5)(2 - 2) = 0$.

(e) Let $Q(x) = g(2x) \cdot f(3x)$. Compute $Q'(2)$.

Solution: Again by the product rule and chain rule, $Q'(x) = 2g'(2x) \cdot f(3x) + 3f'(3x) \cdot g(2x)$ so $Q'(1) = 2g'(2\cdot 2) \cdot f(3\cdot 2) + 3f'(3\cdot 2) \cdot g(2\cdot 2) = 2g'(4) \cdot f(6) + 3f'(6) \cdot g(4) = 2 \cdot 6 \cdot 0 + 3 \cdot 3 \cdot 2 = 18$.

(f) Let $V(x) = f(2 + g(x - 2))$. Compute $V'(2)$.

Solution: Again, by the chain rule, $V'(x) = f'(2 + g(x - 2)) \cdot g'(x - 2)$, so $V'(2) = f'(g(0)) \cdot g'(0) = f'(5) \cdot g'(0) = 3 \cdot 2 = 6$.

(g) Let $W(x) = g(x - f(x))$. Compute $W'(5)$.

Solution: Again by the chain rule, $W'(x) = g'(x - f(x)) \cdot (1 - f'(x)$, so $W'(5) = g'(5 - f(5)) \cdot (1 - f'(5)) = 2 \cdot -2 = -4$.